King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 601
Midterm Exam– Term 241
Monday, November 18, 2024
Allowed Time: 120 minutes

Name: ______________________________

ID #: ____________________

Section #: __________ Serial Number: __________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justifications !

Question #	**Grade**	**Maximum Points**
1		**11**
2		**15**
3		**10**
4		**12**
5		**06**
6		**09**
7		**14**
8		**13**
Total:		**90**

Exercise 1:(11)

A-(05) Let X_1 and X_2 be two independent exponential random variables with respective means $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$. Prove that

$$P\{X_1 < X_2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

B-(06) Let $N(t)$, $t \geq 0$ be a Poisson process with parameter λ.

Show that $\lim_{t \longrightarrow \infty} \frac{N(t)}{t} = \lambda$ a.s.

Hint: You may use the Strong law of large numbers : $\lim_{n \longrightarrow \infty} \frac{N(n)}{n} = \lambda$ a.s.

Exercise 2:(15)

A-(04) A particle moves among states $0, 1$ and 2 according to a Markov process with the following transition probability matrix.

$$P = \begin{pmatrix} 0.6 & a & 0.1 \\ b & 0.3 & 0.4 \\ 0.3 & 0.2 & c \end{pmatrix}$$

where a, b and c are real numbers.

Let X_n be the position of the particle after the $n-$th move. Suppose that at the beginning, the particle is in state 1. Determine the probability $P[X_2 = k]$ where $k = 0, 1, 2$.

B-(11) Consider a birth process and let $\lambda_i = \nu_i \, p_{i(i+1)}$ be the rate at which a birth occurs when the process is in state i and $p_{i(i+1)}$ its corresponding probability, and ν_i is the rate at which the process makes a transition when in state i into j. Let also $q_{ij} = \nu_i \, P_{ij}$ be the instantaneous transition rates, where P_{ij} is the transition probabilities from state i into state j.

Given that the kolmogorov's forward equations are given by:

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - \nu_j \, P_{ij}(t). \tag{a}$$

i)-(05) Find the forward equations for the pure birth process.
(Discuss the cases $i = j$ and $j \geq i + 1$.)

ii)-(06) Given that $P_{ii}(0) = 1$, deduce from the result of question (i) that

$$\begin{cases} P_{ij}(t) = \lambda_{j-1} \, e^{-\lambda_j \, t} \displaystyle\int_0^t e^{\lambda_j \, s} \, P_{i,j-1}(s) \, ds, \ j \geq i + 1. \\ P_{ii}(t) = e^{-\lambda_i \, t}, \ i \geq 0. \end{cases} \tag{b}$$

Exercise 3: (10)
A-(06)Let $Y(t),\ t \geq 0$ be a Brownian motion process with drift coefficient μ and variance parameter σ^2. We define a stochastic process $\{X(t), t \geq 0\}$ by: $X(t) = e^{Y(t)}$.
1-(04) Identify the process $X(t)$ and find its expectation.

2-(02) Give some applications of $X(t)$.

B-(04) In the following B_t, $B_1(t)$ and $B_2(t)$, $t \geq 0$ are standard Brownian motions. Show whether or not the following processes are martingales w.r.t the $\sigma-$algebra $\{\mathcal{F}_t\}$.
1- $M_t := B_t^2 - t$

2- $X_t = B_1(t)B_2(t)$, where $(B_1(t), B_2(t))$ is a 2-dimensional Brownian motion.

Exercise 4: (12)

A-(06) Let $t_0^n < t_1^n < \cdots < t_n^n = T$, where $t_i^n = i\frac{T}{n}$, be a partition of the interval $[0, T]$ into n equal parts. We denote by

$$\Delta_i^n B = B(t_{i+1}^n) - B(t_i^n) \tag{c}$$

the corresponding increments of the Brownian motion $B(t)$.

Show that

$$\lim_{n \longrightarrow +\infty} \sum_{i=0}^{n-1} (\Delta_i^n B)^2 = T, \quad in \;\; L^2. \tag{d}$$

B-(06)

i)-(02) Give major reasons behind the failure of classical integrations methods, when applied to stochastic processes.

ii)-(04) Provide two main differences between the Riemann integrals and Itô integrals.

Exercise 5: (06)
Let S_t be the price of a stock at time t. Suppose that stock price is modelled as a geometric Brownian motion $S_t = S_0\, e^{\mu\, t + \sigma\, B_t}$, where B_t is a standard Brownian motion.

1- Suppose that the parameter values are $\mu = 0.055$ and $\sigma = 0.07$.
Given that $S_5 = 100$, find the probability that S_{10} is greater than 150. (you may express the result as $\Phi(x)$, where Φ is the standard Normal distribution function and x a real number.)

Exercise 6:(09)

A- (04)Let $\{B_t,\, t \geq 0\}$ be a standard Brownian motion and let f be a function having continuous derivative on $[a, b]$.

a)- Define the Itô stochastic integral on $[a, b]$.

b)- Find the expectation of the Itô integral.

B-(05) Prove the Itô isometry.

Exercise 7:(14)
In the following $B(t)$, $B_1(t)$, $B_2(t)$ and $B_3(t)$ are standard Brownian motions.
A-(09) Let $X_t = e^{B_1(t)}\cos(B_2(t)),\ Y_t = e^{B_1(t)}\sin(B_2(t)),\ Z_t = e^{B_1(t)}$.
Write down dX_t, dY_t and dZ_t as $\alpha\, Y_t + \beta\, X_t + aZ_t,\ \ \gamma\, X_t + \eta\, Y_t + bZ_t$ and $\delta\, Z_t + cX_t + d\,Y_t$ respectively. (Here α, β, γ, η, δ, a, b , c, d) are coefficients to be determined !

B-(05) Write the following stochastic processes X_t on the standard form

$$dX_t = u\,dt + v\,dB_t,$$

for suitable choices of u and v.
i)-(03)$X_t = \Big(B_1(t) + B_2(t) + B_3(t),\ B_2^2(t) - B_1(t)B_3(t)\Big)$, where (B_1, B_2, B_3) is 3-dimensional.

ii)-(02) $X_t = \frac{B_t}{1+t}$

Exercise 8: (13)
Let $\{B_t,\, t \geq 0\}$ be a standard Brownian motion defined on $[0,T]$.
Part A(07):
1-(02) Prove that $\int_0^T B_t\, dt = \int_0^T (T-t)\, dB_t$.

2-(02) Write B_T^2 in integral form.

3-(03) Prove that $B_T^3 = \int_0^T X_t\, dB_t$, where X_t is a stochastic process to be determined. (**Hint:** You may find first $d(B_t^3)$ then use the results of question 1)

Part B(06):
Write the following stochastic processes in integral forms:

1-(03) e^{B_T}, by considering the Itô exponential.

2-(03) $\sin(B_T)$