King Fahd University of Petroleum and Minerals

Department of Mathematics

Math 601
Final Exam– Term 241
Wednesday, December 18, 2024
Allowed Time: 120 minutes

Name: ______________________________

ID #: ____________________

Section #: ___________ Serial Number: ___________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justifications !

Question #	Grade	Maximum Points
1		11
2		11
3		15
4		12
5		14
6		10
7		15
8		12
Total:		100

Exercise 1:(11)

A. (07)i)- Let $X : \Omega \longrightarrow \mathbb{R}^n$ be a random variable such that

$$\mathbb{E}[\mid X \mid^p] < \infty, \quad 0 < p < \infty. \tag{a}$$

Prove Chebychev's inequality

$$P[\mid X \mid \geq \lambda] \leq \frac{1}{\lambda^p} \mathbb{E}[\mid X \mid^p] \quad for \;\; all \;\; \lambda \geq 0. \tag{b}$$

Hint: Use the subset A of Ω given by: $A = \{\omega : \mid X \mid \geq \lambda\}$.

ii)- Suppose there exists $k > 0$ such that

$$M = \mathbb{E}[\exp(k \mid X \mid)] < \infty. \tag{c}$$

Prove that

$$P[\mid X \mid \geq \lambda] \leq M\, e^{-k\lambda} \quad for \;\; all \;\; \lambda \geq 0. \tag{d}$$

Hint: Use the result found in i).

B-(04) Suppose cars entering a parking lot follow a Poisson process with rate $\lambda = 5$ per minute. Further, suppose the probability that a driver is female is 0.6. Find the probability that exactly 3 cars driven by females will enter the lot in the next 2 minutes.

Exercise 2:(11)

Let X and Y be two random variables with joint density function

$$f(x,y) = x(y-x)\,e^{-y}, \quad 0 < x \leq y < \infty. \tag{e}$$

1-(06). Find the conditional probability densities functions $f_{X|Y}$ and $f_{Y|X}$.

2-(05). Deduce from 1)- the conditional expectation of X given Y, $\mathbb{E}(X \mid Y)$.

Exercise 3: (15)

A-(08) Consider the standard Brownian motion $\{B_t,\, t \geq 0\}$.

a)- Find $\mathbb{E}(\mid B_t - B_s \mid^2)$, for arbitrary $\;s, t \geq 0$.

b)- Given that $\displaystyle\int_{\mathbb{R}} e^{\frac{-(x - i\,\lambda\,t)^2}{2t}}\,dx = \sqrt{2\,\pi\,t}$, compute the characteristic function of B_t.

c)- Deduce from b) $\;\mathbb{E}(B_t^4)$.

B-(07). A stochastic process $X_t(.) : \Omega \longrightarrow \mathbb{R}$ is continuous in mean square if $\mathbb{E}[X_t^2] < \infty$ for all t and

$$\lim_{s \to t} \mathbb{E}[(X_s - X_t)^2] = 0 \quad \text{for all } t \geq 0.$$

1- Prove that the Brownian motion B_t is continuous in mean square.

2- Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be Lipschitz continuous function.
Prove that the process $Y_t := f(B_t)$ is continuous in mean square.

Exercise 4: (12)
The Black-Scholes-Merton model for growth with uncertain rate of return, is the value of \$1 after time t, invested in a saving account. It is described by the stochastic differential equation:

$$\forall t \geq 0, X_t = x_0 + \int_0^t bX_s\, ds + \int_0^t \sigma\, X_s\, dB_s, \tag{f}$$

where $\{B_t, t \geq 0\}$ is a 1-dimensional Brownian motion, $b, \sigma \in \mathbb{R}$ and $x_0 \in \mathbb{R}$.

1- Give the type of the SDE (f).

2- Asuume that $\mathbb{E}(|\, x_0\,|^2) < \infty$, show that the SDE (f) admits a unique solution.

3-Show that the solution of the SDE (f) is given by:

$$X_t = x_0\, \exp\Big((b - \frac{\sigma^2}{2})t + \sigma\, B_t\Big). \tag{g}$$

4- Give the type of the stochastic process X_t given by (g) and state some of its applications.

<u>Exercise 5:</u> (14)
Let $X_t = B_t$ be a 1-dimensional Brownian motion and

$$g(t,x) = e^{ix} = (\cos x, \sin x) \in \mathbb{R}^2 \ \ for \ \ x \in \mathbb{R}. \tag{h}$$

Then $Y(t) = g(t, X_t)$ is again an Itô process.
1-(04). Prove that the coordinates Y_1, Y_2 of Y satisfies two SDEs in terms of B_t.

2-(06). Show that the process $Y = (Y_1, Y_2)$, called Brownian motion on the unit circle, satisfy the SDE

$$dY(t) = \alpha\, Y(t)\, dt + K\, Y(t)\, dB_t, \tag{i}$$

where $\alpha \in \mathbb{R}$ and K an appropriate matrix.

3-(04). Solve the SDE (i).

<u>Exercise 6:</u>(10)

To model the spot freight rate in shipping, J.Tvedt(1995) used the geometric mean reversion process X_t which is defined as the solution of the stochastic differential equation

$$dX_t = \kappa\,(\alpha - \log\, X_t)\, X_t\, dt + \sigma\, X_t\, dB_t; \;\; X_0 = x > 0, \tag{j}$$

where κ, α, σ and x are positive constants.
Find the solution X_t of the stochastic differential equation (j).
Hint: You may use the substitution $Y_t = \log X_t$ to transform the equation (j) into a linear stochastic differential equation for Y_t.

Exercise 7:(15)
Consider the Ornstein-Uhlenbeck process $\{X(t), t \geq 0\}$ given by the SDE:

$$dX_t = \kappa(m - X_t)\, dt + \sigma\, dB_t, \tag{k}$$

where $\{B_t,\, t \geq 0\}$ is a standard Brownian motion and κ, m and σ are positive constants.
1-(02). Let $Y_t = X_t - m$. Verify that the process Y_t satisfy a given SDE in Y_t.

2-(05). The process Y_t is seen to have a drift at an exponential rate κ. Use the change of variable $Z_t = e^{\kappa t} Y_t$ to write an SDE in the variable Z_t. Find the solution of the obtained SDE over $[s, t]$, $0 \leq s < t$.

3-i)-(04). Deduce the solution of the SDE in Y_t obtained in 1) over $[s, t]$.

ii)-(04). Deduce the solution of the SDE (k).

Exercise 8: (12)

Let $\{B_t,\, t \geq 0\}$ be a standard Brownian motion and suppose that $\{X(t), t \geq 0\}$ is an Ornstein-Uhlenbeck process given by the SDE:

$$dX_t = -a(X_t - g(t))\, dt + \sigma(t)\, dB_t, \tag{l}$$

where g and σ are deterministic function of time and $a \in \mathbb{R}$

1-(06). If $Y_t = e^{X_t + ct}$, $c \in \mathbb{R}$, then prove that $\frac{dY_t}{Y_t}$ can be written as

$$\frac{dY_t}{Y_t} = h(t, X_t)\, dt + \alpha(t)\, dB_t, \tag{m}$$

where h and α are suitable functions to be detrmined !

2-(06). Deduce from the previous question that $\frac{dY_t}{Y_t}$ can be also written as

$$\frac{dY_t}{Y_t} = \psi(t, Y_t)\, dt + \alpha(t)\, dB_t, \tag{n}$$

where ψ is a suitable function to be determined !

Name.