KFUPM-DEPARTMENT OF MATHEMATICS-MATH 645-EXAM I-TERM 231

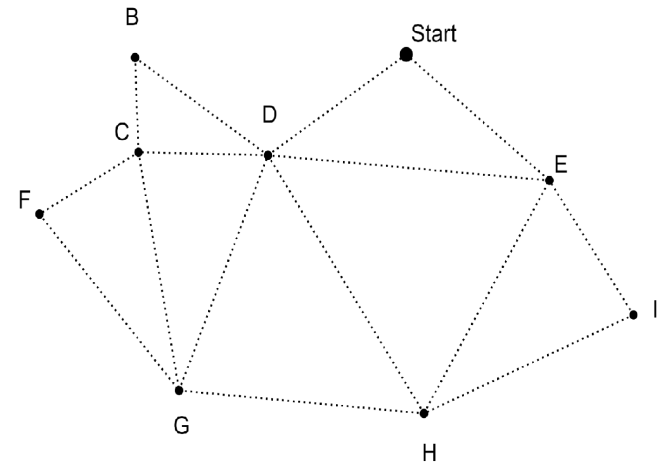
MATH 645: EXAM I, TERM (231), OCTOBER 11, 2023

EXAM I- MATH 645
Duration: 150 mn

Student Name:

ID:

Exercise 1. Consider the following graph:

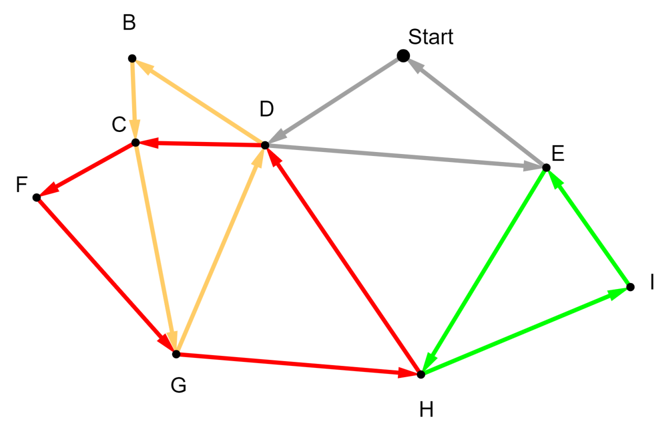


Graph G

(1) Explain why G is Eulerian.
(2) Decompose G into cycles.
(3) Find an Euler tour of G starting from vertex "Start".

Solution. **1**. It is clear that the degree of every vertex of G is even. So G is Eulerian.

2.A possible decomposition into cycle:



3. An Euler tour starting from the vertex Start:

$$S - D - B - C - G - D - C - F - G - H - D - E - H - I - E - S$$

□

Exercise 2. Let $n \geq 4$, and S be a set of cardinality n. We let $\mathbf{PG}[n, 2]$ be the simple graph with vertices the subsets of S of cardinality 2, and adjacency relation given by: A, B are adjacent if and only if $A \cap B = \emptyset$.

(1) Show that $\mathbf{PG}[n,2]$ is $\frac{(n-2)(n-3)}{2}$-regular and has $\frac{n(n-1)}{2}$ vertices.
(2) Evaluate the number of edges of $\mathbf{PG}[n,2]$.
(3) Show that if $n=4$, then $\mathbf{PG}[n,2]$ is isomorphic to $3K_2$ (three copies of K_2).

IN ALL THE REMAINING QUESTIONS, WE WILL ASSUME THAT $n \geq 5$.

(4) Show that $\mathbf{PG}[n,2]$ is connected and $d(u,v)=2$, for any two distinct non-adjacent vertices.
(5) Show that $\mathbf{PG}[n,2]$ is Eulerian if and only if either $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$.
(6) Show that $\mathbf{PG}[n,2]$ contains a triangle if and only if $n \geq 6$.
(7) Show that $\mathbf{PG}[n,2]$ contains a quadrilateral (a 4-cycle) if and only if $n \geq 6$.

Solution.

1. Let A be a vertex of $\mathbf{PG}[n,2]$. The neighborhood of A is given by:

$$\mathbf{N}(A) = \{B \subseteq [n] : A \cap B = \emptyset\} = \{B : B \subseteq [n] \setminus A \text{ and } |B| = 2\} = \mathcal{P}([n] \setminus A, 2),$$

where $\mathcal{P}([n] \setminus A, 2)$ is the set of all subsets of $[n] \setminus A$ with size 2. So $d(A) = \binom{n-2}{2} = \frac{(n-2)!}{(n-4)!2!} = \frac{(n-2)(n-3)}{2}$.

It follows that $\mathbf{PG}[n,2]$ is $\frac{(n-2)(n-3)}{2}$-regular.

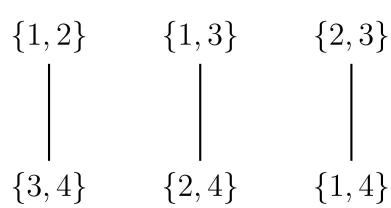
2. The vertex set of $\mathbf{PG}[n,2]$ is $\mathcal{P}([n],2)$. By Euler's Sum of Degrees Theorem, we have

$$\sum_{A \in \mathcal{P}([n],2)} d(A) = 2m,$$

where m is the number of edges of $\mathbf{PG}[n,2]$.

Hence $|\mathcal{P}([n],2)| \times \frac{(n-2)(n-3)}{2} = 2m$, that is $\binom{n}{2} \times \frac{(n-2)(n-3)}{2} = 2m$, and consequently, $m = \dfrac{n(n-1)(n-2)(n-3)}{8}$.

3. For $n=4$, the graph $\mathbf{PG}[4,2]$ looks like:



which is, clearly, isomorphic to $3K_2$.

4. Let A, B be two distinct nonadjacent vertices. Then $|A \cap B| = 1$. So $|A \cup B = 2+2-1 = 3$. Consequently $\big|[n] \setminus (A \cup B)\big| \geq 2$. Taking any subset C of $[n] \setminus (A \cup B)$ of size 2, C is adjacent to both A and B. It follows that $\mathbf{PG}[n,2]$ is connected and $d(A,B)=2$, for all $A \neq B$, nonadjacent.

5. As $\mathbf{PG}[n,2]$ is $\frac{(n-2)(n-3)}{2}$-regular, it is Eulerian iff $\frac{(n-2)(n-3)}{2}$ is even, which is equivalent to 4 divides $(n-2)(n-3)$. But as $\gcd(n-2, n-3) = 1$, $\mathbf{PG}[n,2]$ is

Eulerian iff either 4 divides $n-2$ or 4 divides $n-3$, which means $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$.

6. A triangle (A, B, C) in $\mathbf{PG}[n, 2]$ means $\emptyset = A \cap B = A \cap C = B \cap C$. So by Addition Rule, $\left|A \cup B \cup C\right| = \left|A\right| + \left|B\right| + \left|C\right| = 6$. Hence $\mathbf{PG}[n, 2]$ contains a triangke if and only if $n \geq 6$.

7. We know that $\mathbf{PG}[5, 2]$ (the Petersen Graph) is C_4-free.

Now, assume that $n \geq 6$, Let $A \neq B$ two 2-sets such that $\left|A \cap B\right| = 1$. So $\left|A \cup B\right| = 3$. As $n \geq 6$, there exist $\binom{n-3}{2} \geq 3$ common adjacent vertices to A and B. So $\mathbf{PG}[n, 2]$ contains a quadrilateral.

$\square$

Exercise 3. Let G be a finite group of cardinality n and identity element 1. Let S be a subset of G such that $1 \notin S$ and $S^{-1} = S$ (i.e. $s \in S \Longrightarrow s^{-1} \in S$).

We denote by $\mathrm{Cay}(G : S)$ (Cayley graph) the simple graph with set of vertices G and adjacency relation defined by: x, y are adjacent if and only if $y = sx$, for some $s \in S$.

(1) Show that $\mathrm{Cay}(G : S)$ is $\left|S\right|$-regular.

(2) For $g \in G$, let

$$\begin{array}{rccc} \varphi_g\colon & G & \longrightarrow & G \\ & x & \longmapsto & xg \end{array}$$

Show that φ_g is an automorphism of $\mathrm{Cay}(G : S)$.

Show that $\mathrm{Cay}(G : S)$ is vertex-transitive.

(3) Show that $\mathrm{Cay}(G : S)$ is connected if and only if S generates G.

(4) Show that if G is Abelian and $\left|S\right| \geq 3$, then $\mathrm{Cay}(G : S)$ contains a 4-cycle.
(5) Show that the Petersen Graph is not a Cayley graph. (Recall that if G is a group of order 10, then either $G \simeq \mathbb{Z}_{10}$ or $G \simeq D_5$, where D_5 is the group generated by two elements r and s, with $r^5 = 1 = s^2$ (the order of r is 5 and the order of s is 2) and $srs = s^{-1}$).

Solution.

(1) Let v be a vertex of $\Gamma = \mathrm{Cay}(G : S)$. Consider the function

$$\begin{array}{rcccl} \gamma & : & S & \longrightarrow & N(v) \\ & & s & \longmapsto & sv \end{array}$$

Clearly γ is a bijection. Hence $\left|S\right| = \left|N(v)\right| = d(v)$. Thus Γ is $\left|S\right|$-regular.

(2) It is clear that φ_g is a bijection. For $x, y \in G$, we have:

$$\begin{array}{rcl} xy \in E(\Gamma) & \Longleftrightarrow & y = sx \text{ for some } s \in S \\ & \Longleftrightarrow & yg = sxg \\ & \Longleftrightarrow & \varphi_g(y) = s\varphi_g(z) \\ & \Longleftrightarrow & \varphi_g(x)\varphi_g(y) \in E(\Gamma). \end{array}$$

It follows that φ_g is an automorphism of $\mathrm{Cay}(G : S)$.

Let $x \neq y$. Then $y = \varphi_g(x)$, with $g = x^{-1}y$. Thus Γ is vertex transitive.

(3) Suppose that $\Gamma = \mathrm{Cay}(G : S)$ is connected. Then for each $x \in G$, there is a path from 1 to x.

Let $p = (1, x_1, \ldots, x_k = x)$ be such a path. Then $x_1 = s_1.1$, for some $s_1 \in S, \ldots, x_k = s_k s_{k-1} \ldots s_1$, for some $s_1, s_2 \ldots, s_k \in S$. Thus S generates G.

Conversely, suppose that S generates G, then for every $x \in G$, $x = s_1 s_2 \ldots s_k$ for some $s_1, s_2, \ldots, s_k \in S \cup S^{-1} = S$. Hence

$$W_x = (1, s_k, s_{k-1}s_k, \ldots, x = s_1 s_2 \ldots s_k)$$

is a walk from 1 to x. Thus, for all $x \neq y$ in G, concatenating the walks W_x and W_y at the vertex 1, we obtain a walk from x to y, showing that the graph Γ is connected.

(4) Assume G is an Abelian group and $\left|S\right| \geq 3$. Let $s \in S$, then there exists $t \in S$ such that $t \notin \{s, s^{-1}\}$.

So

$$\begin{array}{ccc} t & \text{———} & st = ts \\ | & & | \\ 1 & \text{———} & s \end{array}$$

is a 4-cycle of Γ.

(5) Let **PG** be the Petersen graph. We know that **PG** satisfies the following properties.

- It has $n = 10$ vertices.
- It is 3-regular.

- It does not contain a 4-cycle.
- It contains a 5-cycle.
- It is vertex transitive.

We will show that it is not isomorphic to a Cayley graph.

Indeed, suppose that $\mathbf{PG} = \Gamma = \mathrm{Cay}(G : S)$, for some group G and some Cayley set S. Then $|G| = 10$. According the above questions, the following properties hold:

- $|S| = 3$ (as $\mathbf{PG}$ is 3-regular.
- G is not Abelian (as $\mathbf{PG}$ has no 4-cycle).

From Group Theory, the groups of order 10 are exactly $\mathbb{Z}/10\mathbb{Z}$ and the Dihedral group $\mathcal{D}_5 =< r, s >$ such that $\mathrm{o}(r) = 5$, $\mathrm{o}(s) = 2$, and $srs = r^{-1}$. We deduce that $G = \mathcal{D}_5$.

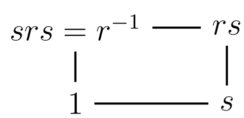
The descriptive list of elements of $\mathcal{D}_5$ is

$$\mathcal{D}_5 = \left\{1, r, r^2, r^3, r^4, s, rs, r^2s, r^3s, r^4s\right\}.$$

Note that the rotations r, r^2, r^3 and r^4 play the same role in $\mathcal{D}_5$ and the reflections s, rs, r^2s, r^3s, r^4s play, also, the same role; in the sense that if $r_1 \in \{r, r^2, r^3, r^4\}$ and $s_1 \in \{s, rs, r^2s, r^3s, r^4s\}$, then $\mathcal{D}_5 =< r_1, s_1 >$.

We consider two cases.

Case 1: If S contains r, then $S = \{r, r^{-1}, s\}$. In this case



is a 4-cycle of $\mathbf{PG}$, a contradiction.

Case 2: If S does not contain r. Then S is a subset of $\{s, rs, r^2s, r^3s, r^4s\}$ of cardinality 3.

As $\mathbf{PG}$ contains a 5-cycle, there exist $t_1, t_2, t_3, t_4, t_5 \in S$ such that

$$t_1t_2t_3t_4t_5 = 1.$$

As every $t_i^2 = 1$ for every i, we deduce that $t_1 = (t_2t_3)(t_4t_5)$.

Note that if i is a nonnegative integer, then as $srs = srs^{-1} = r^{-1}$, we deduce that $sr^is = (srs^{-1})^i = r^{-i}$. So if $t, t' \in S$, then $t = r^is$ and $t' = r^js$ for some nonnegative integers i, j, and

$$tt' = r^isr^js = r^i(sr^js) = r^ir^{-j} = r^{i-j}.$$

Thus

$$t_1 = (t_2t_3)(t_4t_5) = r^k,$$

for some nonnegative integer k, which is impossible, as t_1 is of order 2 and r^k is either 1 or of order 5.

As a result, $\mathbf{PG}$ is a vertex transitive graph that is not a Cayley graph.

□