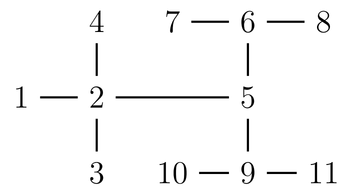
EXAM II- MATH 645
Duration: 120 mn

Student Name:

ID:

Exercise 1. Find the Prüfer code of the following labelled tree.



Solution. For a finite labeled tree T (on $[n]$, with $n \geq 3$), we denote by $T_1 = T$, $L_1 = \mathcal{L}(T_1)$ (the set of all leaves of T_1, $\ell_1 = \min(L_1)$, s_1 the neighbor of ℓ_1 in T_1, and the word $C_1 = s_1$ (over the alphabet $[n]$).

Recursively, for i from 1 to $n-3$, we define $T_{i+1} = T_i - \ell_i$, $L_{i+1} = \mathcal{L}(T_{i+1})$, $\ell_{i+1} = \min(L_{i+1})$, s_{i+1} the neighbor of ℓ_{i+1} in T_{i+1}, and the word $C_{i+1} = C_i s_{i+1}$ (as concatenation of words).

The Prüfer code of T is C_{n-2}.

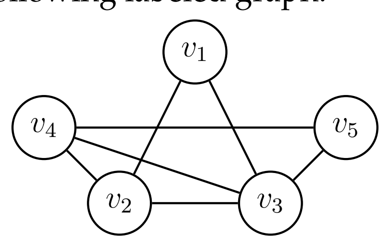
For the given tree, we have the following table.

i	Elements of L_i	ℓ_i	s_i	C_i
1	$1, 3, 4, 7, 8, 10, 11$	1	2	2
2	$3, 4, 7, 8, 10, 11$	3	2	22
3	$4, 7, 8, 10, 11$	4	2	222
4	$2, 7, 8, 10, 11$	2	5	2225
5	$7, 8, 10, 11$	7	6	22256
6	$8, 10, 11$	8	6	222566
7	$6, 10, 11$	6	5	2225665
8	$5, 10, 11$	5	9	22256659
9	$10, 11$	10	9	222566599

It follows that the Prüfer code of T is $C = 222566599$.

□

Exercise 2. Consider the following labeled graph:



Graph G

(1) Use "Greedy Algorithm" to color G.
(2) Find $\chi(G)$.

Solution. The greedy algorithm of coloring the vertices of a graph G consists of the following steps.

Step 1 : Choose an arbitrary "order labeling" of the vertices of G: $v_1, v_2, \ldots, v_n$.

Step 2 : Define a function $f : V \longrightarrow \mathbb{N} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots\}$ by setting $f(v_1) = 1$, and recursively, for $i \geq 2$, if $W_i = N_G(v_i) \cap \{v_1, \ldots, v_{i-1}\}$, then

$$f(v_i) = \min\left(\mathbb{N} \setminus f(W_i)\right).$$

As each vertex has at most $\Delta(G)$ earlier neighbours, the "greedy colouring" uses at most $\Delta(G) + 1$ colors.

The following table illustrate Greedy algorithm for the given graph.

i	v_i	$W_i = N_G(v_i) \cap \{v_1, \ldots, v_{i-1}\}$	$f(W_i)$	$f(v_i)$
1	v_1	$\emptyset$	$\emptyset$	$\mathbf{1}$
2	v_2	$\{v_1\}$	$\{\mathbf{1}\}$	$\mathbf{2}$
3	v_3	$\{v_1, v_2\}$	$\{\mathbf{1}, \mathbf{2}\}$	$\mathbf{3}$
4	v_4	$\{v_2, v_3\}$	$\{\mathbf{2}, \mathbf{3}\}$	$\mathbf{1}$
5	v_5	$\{v_3, v_4\}$	$\{\mathbf{1}, \mathbf{3}\}$	$\mathbf{2}$

Hence the Greedy coloring uses 3 colors, and consequently $\chi(G) \leq 3$. As in addition G is not bipartite(it contains a 3-cycle) and not empty and not empty, we deduce that $\chi(G) \geq 3$. As a result, $\chi(G) = 3$. □

Exercise 3. Let T be a tree with at least 2 vertices and ℓ be the number of leaves of T.

(1) Show that $\ell = 2 + \sum_{\substack{v \in V(T) \\ d(v) \geq 2}} (d(v) - 2)$. Deduce that T has at least 2 leaves.

(2) Show that if T is a path if and only if it has exactly 2 leaves.

(3) Show that if G is a connected simple graph with at least 2 vertices, then there exist two distinct vertices u, v of G such that $G - \{u, v\}$ is connected.

Solution. (1) As T is a tree, we have $|E(T)| = |V(T)| - 1$. So, by the Fundamental Theorem of Graph Theory, we deduce that:

$$\begin{aligned}
2 &= 2|V(T)| - 2|E(T)| \\
&= 2|V(T)| - \sum_{v\in V(T)} d(v) \\
&= \sum_{v\in V(T)} (2-d(v)) \\
&= \sum_{\substack{v\in V(T)\\ d(v)=1}} (2-d(v)) + \sum_{\substack{v\in V(T)\\ d(v)\geq 2}} (2-d(v)) \\
&= \sum_{\substack{v\in V(T)\\ d(v)=1}} 1 + \sum_{\substack{v\in V(T)\\ d(v)\geq 2}} (2-d(v)) \\
&= \ell + \sum_{\substack{v\in V(T)\\ d(v)\geq 2}} (2-d(v)).
\end{aligned}$$

Therefore $\ell = 2 + \sum_{\substack{v\in V(T)\\ d(v)\geq 2}} (d(v)-2)$. Consequently, $\ell \geq 2$.

(2) Assume T is a path: $T = (v_1, v_2, \ldots, v_n)$. Then $d(v_1) = d(v_2) = 1$ and $d(v_i) = 2$, otherwise. So v_1 and v_2 are the only leaves of T.

Now, suppose that T has exactly two leaves u and v, then according to Question 1., $0 = \sum_{\substack{x\in V(T)\\ d(x)\geq 2}} (d(x)-2)$. Thus $d(x) = 2$, for each $x \in V(T) \setminus \{u, v\}$.

Let $P = (u = u_0, u_1, \ldots, u_k, v)$ be the unique path joining u and v. Suppose that there is a vertex w of T which is not in P. As T is connected, there exists a path $Q = (u = w_0, w_1, \ldots, w_s = w)$ (of course $w_1 = u_1$, as $d(u) = 1$) joining u and w. Let w_i be $w_{i-1} \in P$ the first vertex of Q not in P. Then $i \geq 2$, and $w_{i-1} \notin \{u, v\}$ (as u, v are leaves). So $d(w_{i-1}) = 3$, as it is adjacent to two vertices in P and to w_i (not in P), a contradiction. It follows that all the vertices of T are in the path P, completing the task.

(3) Let G be a connected simple graph with at least 2 vertices and T be a spanning tree of G. Then T has at least 2 leaves u, v. So $T - \{u, v\}$ is a spanning tree of $G - \{u, v\}$. Therefore $G - \{u, v\}$ is connected.

□

Exercise 4. Let T be a tree with $n \geq 3$ vertices and

$$\varphi\colon\ [n] \longrightarrow V(T)$$

be a labeling. Let $\mathrm{P.c}(T) = a_1 \ldots a_{n-2}$ be the Prüfer code of T.

(1) Show that T is a path if and only if for all $i \neq j$ in $[n-2]$, $a_i \neq a_j$.
(2) Show that T is a star if and only if $a_1 = a_2 = \ldots = a_{n-2}$.

Solution.

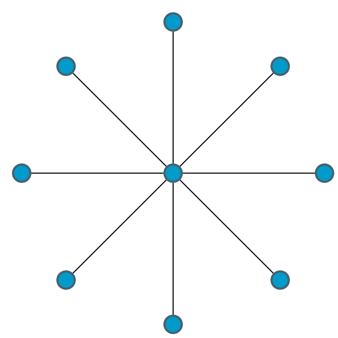
(1) Assume that T is a path, then it has exactly two leaves $u, v \in V(T)$, all the others vertices are of degree 2. So $\varphi^{-1}(u), \varphi^{-1}(v)$ do not appear in $\mathrm{P.c}(T)$ and for each $w \in V(T) - \{u, v\}$, $\varphi^{-1}(w)$ appears once in $\mathrm{P.c}(T)$. It follows that $a_i \neq a_j$, for $i \neq j$.

Conversely, suppose that $a_i \neq a_j$, for $i \neq j$. Thus $|[n] \setminus \{a_1, a_n, \ldots, a_{n-2}\}| = 2$. Thus T has exactly 2 leaves, and consequently T is a path.

(2) Suppose that T is a star with n vertices ($T \sim K_{1,n-1}$). Then T has $n-1$ leaves and a vertex of degree $n-1$. Let $v_1, v_2, \ldots, v_{n-1}$ be the leaves of T and w be the vertex with degree $n-1$. So $\varphi^{-1}(w)$ appears $n-2$ $n-2$ times in $\mathrm{P.c}(T)$. Thus, letting $a = \varphi^{-1}(w)$, we have $\mathrm{P.c}(T) = (a, a, \ldots, a)$.

Conversely, suppose that $\mathrm{P.c}(T) = (a, a, \ldots, a)$. We let $w = \varphi(a)$, then $d(w) = n - 2 + 1 = n - 1$.

For each $x \in V(T) - \{w\}$, $\varphi^{-1}(x)$ does not appear in $\mathrm{P.c}(T)$. We let $x_1, x_2, \ldots, x_{n-1}$ be the leaves of T. As a result, T looks like (for $n = 9$):



$K_{1,8}$

We conclude that T is a star.

□

Exercise 5. Let G be a graph. Show that the following properties hold.

(1) $\chi(G) - 1 \leq \chi(G - v) \leq \chi(G)$ for each vertex v in G.
(2) $\chi(G) - 1 \leq \chi(G - e) \leq \chi(G)$ for each edge e in G
(3) If G contains only one odd cycle as a subgraph, then $\chi(G) = 3$.
(4) If G is not bipartite and has a vertex which is contained in every odd cycle of G, then $\chi(G) = 3$.

Solution.

1. As $G - v$ is a subgraph of G, we get $\chi(G - v) \leq \chi(G)$.
For the left inequality $\chi(G) - 1 \leq \chi(G - v)$; that is $\chi(G) \leq \chi(G - v) + 1$, we let

$k = \chi(G-v)$ and $f : V(G-v) \to \{1, 2, \cdots, k\}$ be a k-colouring of $G-v$. Consider $f' : V(G) \to \{1, 2, \cdots, k, k+1\}$ the mapping defined by:

$$f'(x) = \begin{cases} f(x) & \text{if } x \in V(G-v), \\ k+1 & \text{if } x = v. \end{cases}$$

Clearly, f' is a $(k+1)$-coloring of G. It follows that, $\chi(G) \leq k+1 = \chi(G-v)+1$, as desired.

2. Since $G-e$ is a subgraph of G, we have $\chi(G-e) \leq \chi(G)$. To establish the inequality $\chi(G) - 1 \leq \chi(G-e)$; equivalently, $\chi(G) \leq \chi(G-e)+1$, we let $k = \chi(G-e)$ and f be a k-colouring of $G-e$, where $e = uv$. Define $f' : V(G) \to \{1, 2, \cdots, k, k+1\}$ by

$$f'(x) = \begin{cases} f(x) & \text{if } x \neq v \\ k+1 & \text{if } x = v. \end{cases}$$

It is clear that f' is a $(k+1)$-coloring of G. Therefore$\chi(G) \leq k+1 = \chi(G-e)+1$, as desired.

3. Since G contains an odd cycle, it is neither empty nor bipartite. So $\chi(G) \geq 3$. Let w be any vertex in the unique odd cycle of G. Then $G-w$ contains no odd cycle, and consequently it is a bipartite graph. This leads to $\chi(G-w) \leq 2$. Now, by Question $1.$, we have $\chi(G) \leq \chi(G-w)+1 \leq 3$. Therefore, $\chi(G) = 3$.

4. As G is not bipartite, we deduce that $\chi(G) \geq 3$. Let v be a vertex in G which is contained in every odd cycle in G. Then $G-v$ does not contain an odd cycle. Therefore $G-v$ is bipartite and $\chi(G-v) \leq 2$. Again, using Question $1.$, we obtain $\chi(G) \leq \chi(G-v)+1 \leq 3$. Thus, $\chi(G) = 3$.

□