EXAM I- MATH 645: Combinatorics

(October 17, 2024; 12:30-14:30)

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Exercise 1.1. For a poset $(P, \leq)$, we recall the following definitions:

- A *chain* of $(P, \leq)$ is a subset of P in which every pair of elements is comparable.
- An *antichain* of $(P, \leq)$ is a subset of P in which no two distinct elements are comparable.
- The length of a chain C in P, denoted $\ell(C)$, is defined as $\ell(C) = |C| - 1$.
- The *height* of a poset P, denoted by $\mathrm{ht}(P)$, is defined as
$$\mathrm{ht}(P) = \sup \{\ell(C) : C \text{ is a chain in } P\}.$$
We also denote
$$\kappa(P) = \sup \{|C| : C \text{ is a chain in } P\}.$$
- The *height* of an element $x \in P$, denoted by $\mathrm{ht}(x)$, is the supremum of the lengths of all chains in P that have x as their greatest element.
- The *width* of a poset P, denoted by $\alpha(P)$, is given by
$$\alpha(P) = \sup \{|A| : A \text{ is an antichain in } P\}.$$

We denote by

$$\begin{aligned} \gamma(P) &= \min (\{n : \{A_1, A_2, \ldots, A_n\} \text{ is a partition of } P \text{ into antichains of } P\}) \\ \theta(P) &= \min (\{n : \{C_1, C_2, \ldots, C_n\} \text{ is a partition of } P \text{ into chains of } P\}) \end{aligned}$$

(1) Let S be a nonempty set of size n and $P = \mathbf{2}^S$ the power set of S. Find $\mathrm{ht}(P)$.
(2) Show that for every poset $(P, \leq)$, we have $\kappa(P) \leq \gamma(P)$ and $\alpha(P) \leq \theta(P)$.
(3) Show that for every poset $(P, \leq)$, $\kappa(P) = \gamma(P)$.
(4) Show that for every poset $(P, \leq)$, $\alpha(P) = \theta(P)$.

Solution. □

Exercise 1.2. Recall that if$A$ is a set of size k and B is a set of size n, then the number of surjective functions $f\colon A \to B$ is given by

$$\sigma(k,n) = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i} i^k = \sum_{i=0}^{n}(-1)^{i}\binom{n}{i}(n-i)^k.$$

The number of partitions of a set of size k into n blocks is called the Stirling number of the second kind, it is denoted by $S(k,n)$ (or $\left\{ {k \atop n} \right\}$).

We denote by $\mathrm{Surj}([k],[n])$ the set of all surjections from the set $[k]$ onto the set $[n]$ and $\mathcal{S}([k],n)$ the set of all partitions of $[k]$ into n blocks.

(1) We define $\Psi : \mathrm{Surj}([k],[n]) \longrightarrow \mathcal{S}([k],n)$, by assigning to $f \in \mathrm{Surj}([k],[n])$, the partition $\mathcal{P} = \{f^{-1}(\{1\}),\ldots,f^{-1}(\{n\})\} \in \mathcal{S}([k],n)$. Show that

$$|\mathrm{Surj}([k],[n])| = \sum_{\mathcal{P}\in\mathcal{S}([k],n)} \left|\Psi^{-1}\left(\{\mathcal{P}\}\right)\right|.$$

(2) Let $\mathcal{P} = \{A_1,\ldots,A_n\} \in \mathcal{S}([k],n)$. We define $\theta : [k] \longrightarrow [n]$ by assigning to each element of A_i the value i.

Show that the function

$$\begin{array}{rccc}\gamma : & S_n & \longrightarrow & \Psi^{-1}\left(\{\mathcal{P}\}\right) \\ & \sigma & \longmapsto & \sigma\circ\theta\end{array}$$

is a bijection, and deduce that $\left|\Psi^{-1}\left(\{\mathcal{P}\}\right)\right| = n!$.

(3) Conclude that

$$|S(k,n)| = |\mathcal{S}([k],n)| = \frac{1}{n!}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i} i^k.$$

Solution. □

Exercise 1.3. Find the number of integer solutions of the inequality

$$x_1 + x_2 + x_3 \leq 20,$$

with $x_1 \geq 0, x_2 \geq 1$ and $x_3 \geq 2$.

Solution. □

Exercise 1.4. The set of all derangements of $[n] := \{1, \ldots, n\}$ is denoted by

$$\mathfrak{D}_n := \{\sigma \in S_n \colon \sigma(i) \neq i \text{ for all } i \in [n]\},$$

where S_n is the set of permutations of the set $[n]$.

The cardinality of $\mathfrak{D}_n$, denoted by d_n (or $!n$, known as the "subfactorial" or the n-th rencontres number), satisfies $d_0 = 1$ and $d_1 = 0$.

Use the “Inclusion-Excusion Principle" to show the formula:

$$d_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

Hint: For each $i \in [n]$, define

$$A_i := \{\sigma \in S_n \colon \sigma(i) = i\}.$$

For any subset $Y \subseteq [n]$, show that the set

$$\bigcap_{i \in Y} A_i = \{\sigma \in S_n \colon \sigma(i) = i \text{ for all } i \in Y\}$$

is equipotent to the set of permutations of $[n] \setminus Y$. Use Inclusion-Exclusion Principle to conclude.

Solution. □

Exercise 1.5.

(1) Let $n > k$ be positive integers. Show that
$$\sum_{i=k}^{n} \binom{n}{i} \binom{i}{k} (-1)^{i-k} = 0.$$

(2) Let (b_n) be a sequence of complex numbers. Show that
$$a_n = \sum_{i=0}^{n} \binom{n}{i} b_i \text{ for all } n \Longleftrightarrow b_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} a_i \text{ for all } n.$$

Solution. □

Exercise 1.6. Let n be a positive integer. Evaluate the following sum:

$$\sum_{p+q+r=n} \binom{n}{p, q, r} p 2^{p-1} 3^q 4^r.$$

Solution. $\square$

Exercise 1.7. Let n be an integer greater than or equal to 2. Evaluate the sum:

$$\sum_{k=0}^{n}(k^2+k+1)\binom{n}{k}2^k.$$

Solution. □