EXAM I- MATH 645: Combinatorics

(December 01, 2024 ; 17:30-19:30)

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Exercise 2.1. Solve the following homographic recurrence relations

(1) $x_{n+1} = \dfrac{x_n + 2}{2x_n + 1}$, with $x_0 \neq -\frac{1}{2}$

(2) $x_{n+1} = \dfrac{x_n - 3}{3x_n + 7}$, with $x_0 \neq -\frac{7}{3}$

Solution. □

Exercise 2.2. Let $(a_n)_{n\geq 0}$ be the real sequence whose exponential generating function is

$$A(s) = \frac{e^{-s}}{1+s}.$$

(1) Show that (a_n) satisfies the recurrence relations:

$$a_{n+1} = -(n+1)a_n + (-1)^{n+1} \quad \text{and} \quad a_{n+1} = n(a_{n-1} - a_n) + 2(-1)^{n+1}.$$

(2) Find an explicit formula of a_n.

Solution. The exponential generating function $A(s)$ is defined as:

$$A(s) = \sum_{n=0}^{\infty} a_n \frac{s^n}{n!} = \frac{e^{-s}}{1+s}.$$

It follows that

$$A(s)(1+s) = e^{-s}.$$

Expanding $A(s)(1+s)$, we have:

$$\sum_{n=0}^{\infty} a_n \frac{s^n}{n!} + \sum_{n=0}^{\infty} a_n \frac{s^{n+1}}{n!} = \sum_{n=0}^{\infty} \frac{(-s)^n}{n!}.$$

Reindexing the second summation, we get:

$$\sum_{n=0}^{\infty} a_n \frac{s^n}{n!} + \sum_{n=1}^{\infty} a_{n-1} \frac{s^n}{(n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n s^n}{n!}.$$

Combining terms:

$$a_0 + \sum_{n=1}^{\infty} (a_n + n a_{n-1}) \frac{s^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n s^n}{n!}.$$

Equating coefficients of $\frac{s^n}{n!}$ for $n \geq 1$, it follows that:

$$a_n + n a_{n-1} = (-1)^n, \quad \text{for } n \geq 1.$$

Rewriting, we obtain the recurrence relation:

$$a_n = -n a_{n-1} + (-1)^n, \quad \text{for } n \geq 1.$$

Thus:

$$a_{n+1} = -(n+1)a_n + (-1)^{n+1}, \quad \text{for } n \geq 0.$$

Next, let us derive the recurrence relation $a_{n+1} = n(-a_n + a_{n-1}) = 2(-1)^{n+1}$. From the previously derived recurrence relation:

$$a_n = -na_{n-1} + (-1)^n.$$

Substituting this into the expression for a_{n+1}:

$$\begin{aligned} a_{n+1} &= -(n+1)a_n + (-1)^{n+1} \\ &= -na_n - a_n + (-1)^{n+1} \\ &= -na_n + na_{n-1} + (-1)^{n+1} + (-1)^{n+1} \\ &= n(-a_n + a_{n-1}) + 2(-1)^{n+1}. \end{aligned}$$

To find an explicit formula of a_n, remak that $e^{-s} = \text{egf}((-1)^n)$ and $\frac{1}{1+s} = \text{egf}((-1)^n n!)$. It follows that a_n is the combinatorial product of $(-1)^n$ and $(-1)^n n!$. Therefore

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! (-1)^{n-k} = (-1)^n \left(\sum_{k=0}^{n} \binom{n}{k} (-1)^k k! \right).$$

□

Exercise 2.3. Let Δ_n be the $n \times n$ determinant:

$$\Delta_n = \begin{vmatrix} 5 & 2 & 0 & \cdots & 0 \\ 3 & 5 & 2 & \ddots & \vdots \\ 0 & 3 & 5 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 2 \\ 0 & \cdots & 0 & 3 & 5 \end{vmatrix}.$$

(1) Find a second-order recurrence relation satisfied by Δ_n.
(2) Find an explicit formula for Δ_n.

Solution. Expanding the determinant along the first column, we find:

$$\Delta_{n+2} = 5\Delta_{n+1} - 3 \begin{vmatrix} 2 & 0 & \cdots & 0 & \\ 3 & 5 & 2 & \ddots & \vdots \\ 0 & 3 & 5 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 2 \\ 0 & \cdots & 0 & 3 & 5 \end{vmatrix}.$$

Expanding the second determinant along the first row, we obtain the following linear recurrence relation (LRR):

$$\Delta_{n+2} = 5\Delta_{n+1} - 6\Delta_n.$$

The characteristic equation of this LRR is:

$$r^2 - 5r + 6 = 0,$$

which has roots $r_1 = 2$ and $r_2 = 3$. Thus, the general solution is:

$$\Delta_n = c_1 2^n + c_2 3^n.$$

Using the initial conditions $\Delta_1 = 5$ and $\Delta_2 = 19$, we solve for c_1 and c_2:

$$\Delta_1 = c_1 2^1 + c_2 3^1 = 2c_1 + 3c_2 = 5,$$

$$\Delta_2 = c_1 2^2 + c_2 3^2 = 4c_1 + 9c_2 = 19.$$

Solving this system of equations gives $c_1 = -2$ and $c_2 = 3$. Therefore, the explicit formula for Δ_n is:

$$\Delta_n = 3^{n+1} - 2^{n+1}.$$

□

Exercise 2.4. Let $\mathbf{K}$ be a field. Show the equality

$$\prod_{k=0}^{\infty}\left(1 + X^{2^k}\right) = \frac{1}{1-X} = 1 + X + X^2 + \cdots$$

in $\mathbf{K}[\![X]\!]$.

Solution.

$$\begin{aligned}
\prod_{k=0}^{\infty}\left(1 + X^{2^k}\right) &= \prod_{k=0}^{\infty}\frac{\left(1 + X^{2^k}\right)\left(1 - X^{2^k}\right)}{1 - X^{2^k}} &&= \prod_{k=0}^{\infty}\frac{1 - X^{2^{k+1}}}{1 - X^{2^k}} \\
&= \lim_{n\to\infty}\prod_{k=0}^{n}\frac{1 - X^{2^{k+1}}}{1 - X^{2^k}} &&= \lim_{n\to\infty}\frac{1 - X^{2^{n+1}}}{1 - X} \\
&= \frac{1}{1-X}.
\end{aligned}$$

□

Exercise 2.5. Let $\mathbf{K}[\![X]\!]^{\circ}$ be the group $X\mathbf{K}[\![X]\!] \setminus X^2\mathbf{K}[\![X]\!]$ with operation the composition of formal power series.

Find the reverse (the inverse in the group $(\mathbf{K}[\![X]\!]^{\circ}, \circ)$) of the formal power series

$$\alpha = X + X^2 + \cdots$$

Solution. Let $\alpha = X + X^2 + \cdots = \dfrac{X}{1-X}$. Then $\alpha^{\underline{r}}$ is defined by the equation $X = \alpha(\alpha^{\underline{r}}) = \dfrac{\alpha^{\underline{r}}}{1-\alpha^{\underline{r}}}$. It follows that

$$\alpha^{\underline{r}} = \frac{X}{1+X} = \sum_{n=1}^{\infty}(-1)^{n-1}X^n.$$

$\square$

Exercise 2.6. Recall the following formal power series in $\mathbb{C}[\![X]\!]$:

$$\sin(X) := \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}X^{2n+1}, \quad \cos(X) := \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}X^{2n},$$

$$\tan(X) := \frac{\sin(X)}{\cos(X)}, \quad \sinh(X) := \sum_{k=0}^{\infty}\frac{X^{2k+1}}{(2k+1)!},$$

$$\arcsin(X) := \sum_{n=0}^{\infty}\frac{(2n)!}{(2^n n!)^2}\frac{X^{2n+1}}{2n+1}, \quad \arctan(X) := \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}X^{2k+1}.$$

(1) Show that $\exp(\mathrm{i}X) = \cos(X) + \mathrm{i}\sin(X)$ where $\mathrm{i} = \sqrt{-1} \in \mathbb{C}$ (Euler's Formula).
(2) Show that $\sin(2X) = 2\sin(X)\cos(X)$ and $\cos(2X) = \cos^2(X) - \sin^2(X)$ (Hint: Use the previous question and separate real from non-real coefficients).
(3) Show that $\cos^2(X) + \sin^2(X) = 1$ (Pythagorean Identity).
(4) Show that $\sinh(X) = \frac{1}{2}(\exp(X) - \exp(-X))$ and $[\sin(X)]' = \cos(X)$ and $[\cos(X)]' = -\sin(X)$.
(5) Show that $\arctan \circ \tan = X$
(6) Show that $\arctan(X) = \dfrac{\mathrm{i}}{2}\log\left(\dfrac{\mathrm{i}+X}{\mathrm{i}-X}\right)$.

Solution.

(**1**)

$$\begin{aligned}\exp(iX) &= \sum_{n=0}^{\infty}\frac{1}{n!}(iX)^n\\ &= \sum_{n=0}^{\infty}\frac{1}{(2n)!}(iX)^{2n}+\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}(iX)^{2n+1}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}X^{2n}+i\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}X^{2n+1}\\ &= \cos(X)+i\sin(X)\end{aligned}$$

(**2**) As $\exp(i(2X)) = \cos(2X) + i\sin(2X)$, and

$$\begin{aligned}\exp(iX+iX) &= \exp(i(2X)) = (\exp(iX))^2\\ &= (\cos(X)+i\sin(X))^2 = \left[(\cos(X))^2-(\sin(X))^2\right]+i\left[2\sin(X)\cos(X)\right]\end{aligned}$$

we deduce that $\sin(2X) = 2\sin(X)\cos(X)$ and $\cos(2X) = (\cos(X))^2 - (\sin(X))^2$.

(**3**) As $\exp(iX)\exp(-iX) = exp(0) = 1$, we deduce that

$$1 = (\cos(X)+\mathrm{i}\sin(X))(\cos(X)-\mathrm{i}\sin(X)) = (\cos(X))^2+(\sin(X))^2.$$

(**4**) Straightforward.

(**5**) Let us first compute $(\arctan)'$ and $(\tan)'$:

$$\begin{aligned}(\arctan)' &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(2n+1)X^{2n}\\ &= \sum_{n=0}^{\infty}(-1)^nX^{2n}\\ &= \sum_{n=0}^{\infty}(-X^2)^n = \frac{1}{1+X^2}\end{aligned}$$

and

$$(\tan)' = \frac{\cos(X)\cos(X)-\sin(X)(-\sin(X))}{\cos^2(X)} = \frac{1}{\cos^2(X)}.$$

Now, by chain rule, we have:

$$\begin{aligned}(\arctan\circ\tan)' &= (\arctan)'(\tan(X)).(\tan)'(X)\\ &= \frac{1}{1+\tan^2(X)}.\frac{1}{\cos^2(X)}\\ &= \cos^2(X)\frac{1}{\cos^2(X)} = 1.\end{aligned}$$

Thus, as $(\arctan\circ\tan) \in \mathbb{C}[\![X]\!]^\circ$, we deduce that $\arctan\circ\tan = X$. This means that $\arctan$ is the reverse of $\tan$.

(**6**) Recall that

$$\log(1+X)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}X^n \text{ and } \log(1-X)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(-1)^nX^n=-\sum_{n=1}^{\infty}\frac{1}{n}X^n.$$

As,

$$\frac{i}{2}\log\left(\frac{i+X}{i-X}\right)=\frac{i}{2}\log\left(\frac{1-iX}{1+iX}\right)=\frac{i}{2}\left[\log(1-iX)-\log(1+iX)\right],$$

we conclude that

$$\begin{aligned}
\frac{i}{2}\log\left(\frac{i+X}{i-X}\right) &= \frac{i}{2}\left(-\sum_{n=1}^{\infty}\frac{i^n}{n}X^n-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}i^n}{n}X^n\right)\\
&= \frac{i}{2}\left(\sum_{n=1}^{\infty}(-1-(-1)^{n-1})\frac{i^n}{n}X^n\right)\\
&= \frac{i}{2}\left(\sum_{n \text{ is odd}}(-1-(-1)^{n-1})\frac{i^n}{n}X^n\right)\\
&= (-i)\left(\sum_{k=0}^{\infty}\frac{i^{2k+1}}{2k+1}X^{2k+1}\right)\\
&= (-i)\left(\sum_{k=0}^{\infty}i\frac{(-1)^{2k+1}}{2k+1}X^{2k+1}\right)\\
&= \sum_{k=0}^{\infty}\frac{(-1)^{2k+1}}{2k+1}X^{2k+1}=\arctan(X).
\end{aligned}$$

□