FINAL EXAM- MATH 645: Combinatorics

(December 28, 2024 ; 19:00-22:00)

Student Name:

ID:

Preamble

Consider a set D of size $m \geq 1$ and a set of colors $K = \{c_1, c_2, \ldots, c_n\}$. Additionally, consider a group G that acts as a group of permutations on the set D. A coloring of D using colors from K is just a function $f \in K^D$.

Two colorings are considered equivalent if one can be transformed into the other by the action of an element $g \in G$. Formally, two colorings $f_1, f_2 \in K^D$ are equivalent if:

$$f_1 = f_2 g, \quad \text{for some } g \in G.$$

The orbits of this action are referred to as patterns. Recall that, according to Pólya Enumeration Theorem, the number of distinct patterns is given by:

$$\text{Total number of patterns} = \mathbf{Z}_G(|K|, |K|, \ldots, |K|),$$

where $\mathbf{Z}_G$ is the cycle index polynomial of G acting on D.

Recall also that the number of patterns with: a_1 occurrences of color c_1, a_2 occurrences of color c_2, …, a_n occurrences of color c_n (where $\sum_{i=1}^{n} a_i = m = |D|$) is exactly the coefficient of $y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n}$ in the polynomial:

$$\mathbf{Z}_G\left(\sum_{i=1}^{n} y_i, \sum_{i=1}^{n} y_i^2, \ldots, \sum_{i=1}^{n} y_i^m\right).$$

Exercise 1. Let $(P, \leq)$ be a locally finite poset. Recall that the incidence algebra of P is the set $\mathbf{I}(P)$ consisting of all functions $f\colon P \times P \longrightarrow \mathbb{R}$ such that $f(x, y) = 0$ if $x \nleq y$. Let $f, g \in \mathbf{I}(P)$. The convolution (or matrix product) $f * g$ of f and g is defined by:

$$(f * g)(x, y) = \begin{cases} \displaystyle\sum_{x \leq z \leq y} f(x, z)g(z, y) & \text{if } x \leq y \\ 0 & \text{if } x \nleq y \end{cases}$$

Show that the convolution product is associative.

Exercise 2. Let k be a nonnegative integer. We define the function:

$$\sigma_k : \mathbb{N} \longrightarrow \mathbb{R}, \quad n \longmapsto \sum_{d|n} d^k.$$

Recall that the Möbius (arithmetic) function μ is defined as follows:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ is divisible by the square of a prime number,} \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct prime factors.} \end{cases}$$

(1) Explain why
$$n^k = \sum_{d|n} \sigma_k(d)\mu\left(\frac{n}{d}\right).$$
(2) Show that if $\gcd(m, n) = 1$, then $\sigma_k(mn) = \sigma_k(m)\sigma_k(n)$.
(3) Show that if p is a prime number and $n = p^r$, then
$$\sigma_k(p^r) = \begin{cases} \frac{1-p^{k(r+1)}}{1-p^k} & \text{if } k > 0, \\ r + 1 & \text{if } k = 0. \end{cases}$$
(4) Let $n > 1$ be an integer with prime factorization $n = p_1^{r_1} \cdots p_t^{r_t}$, where p_i are primes, r_i are positive integers, and the p_i are mutually distinct. Show that
$$\sigma_k(n) = \begin{cases} \prod_{i=1}^{t} \frac{1-p_i^{k(r_i+1)}}{1-p_i^k} & \text{if } k > 0, \\ \prod_{i=1}^{t}(r_i + 1) & \text{if } k = 0. \end{cases}$$
(5) Using the results above, determine the number of divisors of 1800 and the sum of all divisors of 1800.

Exercise 3. If n is a positive integer, then we denote by $\varphi(n)$ the cardinality of the set $\{i \in [n] : \gcd(i, n) = 1\}$

(1) Show that if d divides n, then

$$|\{a \in [n]\colon\ \gcd(a, n) = d\}| = \varphi\left(\frac{n}{d}\right).$$

(2) Show that for every positive integer n, we have

$$n = \sum_{d|n} \varphi(d).$$

(3) Use Möbius Inversion formula to give an explicit formula of $\varphi(n)$.

Exercise 4. Recall that the number of ways to color the vertices of a regular n-gon with k colors, considering equivalent colorings up to rotation, is given by the formula:

$$\mathbf{N}(n,k) = \frac{1}{n}\sum_{d|n}\varphi(d)k^{n/d},$$

where φ is the Euler totient function.

(1) Write the formula for $\mathbf{N}(n,k)$ when n is a prime number.
(2) If $n = p$ is a prime number, use $\mathbf{N}(p,k)$ to derive Fermat's Little Theorem, which states: $k^p \equiv k \pmod{p}$.
(3) Use $\mathbf{N}(n,k)$ to compute the number of distinct ways to color the edges of a square using k colors, considering equivalent colorings up to rotation.

Exercise 5. Let $G = \langle \rho \rangle$ be the subgroup of S_5 generated by the cycle $\rho = (1\ 2\ 3\ 4\ 5)$. Consider the action of G on $X = [5]$ defined by $\sigma \cdot x = \sigma(x)$, where $\sigma \in G$ and $x \in X$.

(1) Determine the cycle index polynomial for the action of G on X.
(2) Assume the vertices of a regular pentagon are colored using 3 colors. How many distinct color patterns are there under the action of G?
(3) Assume the vertices of a regular pentagon are colored using red, green, and blue. Enumerate the number of distinct patterns under the action of G of the pentagon, where red and green each appear twice, and blue appears once.

Exercise 6. Let $X = [4]$, and let V be the subgroup of S_4 generated by the transpositions $s = (1\ 2)$ and $t = (3\ 4)$.

(1) List all the elements of V.
(2) Consider the action of V on $X = [4]$ defined by $\sigma \cdot x = \sigma(x)$, where $\sigma \in V$ and $x \in X$. Determine the cycle index polynomial corresponding to this action.
(3) Deduce the number of distinct (nonequivalent) colorings of the vertices of a square using k colors, where two colorings are considered equivalent if one can be obtained from the other by applying a permutation $\sigma \in V$.
(4) Assuming the set of colors is $K = \{\text{red}, \text{green}, \text{blue}\}$, enumerate the number of distinct patterns where green appears twice, and red and blue each appear once.