King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 654: Advanced Topics in Algebra

Midterm Exam, Fall Semester 241 (120 minutes)

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Remark: Solve **6** questions including **Q6** and **Q7**. Show **full details.**

Q1. (15 points) Show that

(a) A yoked semiring R is additively cancellative if and only if R is plain.

(b) A semiring R is Gel'fand if and only if $r + c \in U(R)$ for all $c \in U(R)$ and $r \in R$.

(c) A commutative semiring R is a bounded distributive lattice if and only if R is idempotent and 1 is an infinite element.

Q2. (15 points)

(a) Let R be a multiplicatively cancellative semiring. Show that: if $r^n = 1$ for some element $r \in R\backslash\{1\}$ and some $n > 1$, then R is a ring.

(b) Let $(R, +, \cdot)$ be a division semiring and consider the group $R^* = (R\backslash\{0\}, \cdot)$. Show that:

(i) if R^* is not torsion-free, then R is a division ring.

(ii) if R^* is torsion, then $R = \mathbf{B}$ (the Boolean algebra) or R is a field.

Q3. (15 points) Consider the semiring $\mathbb{W}$ of whole numbers. Show that

(a) $\mathbb{W}$ is not a principal ideal semidomain.

(b) $\mathbb{W}$ is Noetherian, but $\mathbb{W}[x]$ is not Noetherian.

(c) There is a prime ideal $\mathfrak{p}$ of $\mathbb{W}$ for which $\mathfrak{p}[x]$ is not a prime ideal of $\mathbb{W}[x]$.

Q4. (15 points) Consider the semirings $B(n, i)$. Show that

(a) $B(n, i)$ is a ring if and only if $i = 0$.

(b) $B(n, i)$ is subtractive if and only if $i \leq 1$.

(c) For $i \geq 6$, the lattice $\mathcal{I}(B(n, i))$ of all ideals of $B(n, i)$ is not modular.

Q5. (15 points) Let R be a semiring. Find

(a) The equalizer of a pair $f, g : M \longrightarrow N$ of morphisms of left R-semimodules.

(b) The coequalizer of a pair $f, g : M \longrightarrow N$ of morphisms of left R-semimodules.

(c) The pullback of a pair $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ of morphisms of of left R-semimodules.

Q6. (20 points) Give and example of the following classes of semirings:

(a) A semiring R that is additively idempotent but 1 is not an infinite element.

(b) A subsemiring S of an additively idempotent semiring R such that 1 is an infinite element in S but not an infinite element in R.

(c) A Gel'fand semiring R in which 1 is not an infinite element.

(d) A left austere semiring.

(e) A maximal ideal of a semiring R that contains all proper ideals of R.

Q7. (20 points) Prove or disprove:

(a) Every ideal of a semiring is kernel of some morphism of semirings.

(b) Any maximal ideal of a semiring is prime.

(c) Every additively cancellative semiring is isomorphic to a subsemiring of a ring.

(d) Every epimorphism of semimodules is surjective.

Bonus. (5 points) Demonstrate that category of left semimodule over a semiring is complete (has all small limits) and cocomplete (has all small colimits).

GOOD LUCK