King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 654: Advanced Topics in Algebra

Final Exam, Fall Semester 241 (180 minutes)

Prof. Jawad Abuihlail

Remark: Solve **6** questions including **Q8** and **Q9**. Show **full details.**

Q1. (15 points) Let R be a semiring.

(a) Show that there exist semirings $R_1, \cdots, R_n$ with $R \simeq R_1 \oplus \cdots \oplus R_n$ if and only if R contains a complete set of orthogonal central idempotents $\{e_1, \cdots, e_n\}$ satisfying

$$e_1 + \cdots + e_n = 1 \text{ and } e_i e_j = \delta_{i,j} \text{ for all } i, j \in \{1, \cdots, n\}.$$

(b) Let $\gamma : R \longrightarrow S$ be a morphism of semirings. If $\{e_1, \cdots, e_n\}$ is a complete set of orthogonal central idempotents of R, then $\{\gamma(e_1), \cdots, \gamma(e_n)\}$ is a complete set of orthogonal central idempotents of S.

(c) Can an integral semidomain contain a complete set of orthogonal central idempotents?

Q2. (15 points) Let R be semiring. We say that a morphism $h : X \longrightarrow Y$ of left R-semimodules is

k-uniform iff $h(x_1) = h(x_2)$ implies $x_1 \equiv_{Ker(h)} x_2$

i-uniform iff $h(X) = \overline{h(X)} := \{y \in Y \mid y \equiv_{h(X)} 0\}$.

Moreover, we say that a sequence $L \xrightarrow{f} M \xrightarrow{g} N$ of left semimodules is **exact** iff $f(L) = Ker(g)$ and g is k-uniform.

Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\ 0 & \longrightarrow & L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 & \longrightarrow & 0 \end{array}$$

be a commutative diagram of let R-semimodules and R-linear maps with exact rows and assume that M_1, M_2 are cancellative and α_2 is i-uniform. Show that if any two of the morphisms α_1, α_2 and α_3 are isomorphisms, then so is the third.

Q3. (15 points) Let R be a semiring and consider $R^{\Delta} := S/D$ where $D = \{(a,a) \mid a \in R\}$ and $S = R \times R$ is the semiring with

$$(a,b)+(c,d) = (a+c, b+d), \ (a,b)(c,d) = (ac+bd, ad+bc) \text{ for all } a,b,c,d \in R.$$

(a) Show that R is non-zeroic if and only if $R^{\Delta} \neq 0/D$.

(b) Assume that R is non-zeroic. Show that R^{Δ} is a ring and the canonical morphism

$$\nu : R \longrightarrow R^{\Delta}, \ r \longmapsto [(r,0)]$$

of semirings is injective if and only if R is additively cancellative.

(c) Assume that R is non-zeroic. Show that we have a functor

$$_R\mathbf{SMod} \longrightarrow \ _{R^{\Delta}}\mathbf{Mod}, \ M \longmapsto M^{\Delta}$$

where the R^{Δ}-action on $M^{\Delta} := (M \times M)/W$ $(W := \{(m,m) \mid m \in M\})$ is given by

$$[(a,b)][(m,n)] := [(am+bn, an+bm)].$$

Q4. (15 points)

(a) Let R be a semiring. Show that a left R-semimodule M is projective if and only if it is a retract of a free left R-semimodule.

(b) Let R be a semiring. Let $\{P_\lambda\}_\Lambda$ be a collection of left R-semimodules. Prove that $\bigoplus_{\lambda \in \Lambda} P_\lambda$ is projective if and only if P_λ is projective for every $\lambda \in \Lambda$.

(c) Give an example of a projective semimodule which is not free.

Q5. (15 points)

(a) Let R be a semiring. Show that if E is an injective left R-semimodule, then every essential monomorphism of left R-semimodules $\alpha : E \longrightarrow E'$ is an isomorphism.

(b) Let R be a semiring. Let $\{E_\lambda\}_\Lambda$ be a collection of left R-semimodules. Prove that $\prod_{\lambda \in \Lambda} E_\lambda$ is injective if and only if E_λ is injective for every $\lambda \in \Lambda$.

(c) Show that $_{\mathbb{B}}\mathbb{B}^{\Lambda}$ injective for any non-empty set Λ.

Q6. (15 points) Let $\gamma : S \longrightarrow R$ be a morphism of semirings.

(a) Show that if M is an injective left S-semimodule, then the left R-semimodule $Hom_S(R, M)$ is an injective left R-semimodule.

(b) Show that ${}_{\mathbb{B}}\mathbb{B}$ injective.

(c) Show that if R is an additively idempotent semiring and M is a left R-semimodule, then $I(M) := Hom_{\mathbb{B}}(R, \mathbb{B}^M)$ is an injective left R-semimodule.

Q7. (15 points) Let R be a semiring. Denote with ${}_R\mathbf{SMod}$ the category of left R-semimodules and with ${}_R\mathbf{CSMod}$ its full subcategory of cancellative left R-semimodules.

(a) Show that the relation $\equiv_{[0]}$ defined on a left R-semimodule M by

$$x \equiv_{[0]} y \text{ iff } \exists\ z \in M \text{ such that } x + z = y + z$$

is a congruence relation which is trivial if and only if M is cancellative.

(b) Show that we have a functor

$$\mathfrak{c} :{}_R \mathbf{SMod} \longrightarrow {}_R \mathbf{CSMod},\ M \longrightarrow M/\equiv_{[0]}.$$

(c) Show that $\mathfrak{c} :{}_R \mathbf{SMod} \longrightarrow {}_R \mathbf{CSMod}$ is left adjoint to the embedding functor ${}_R\mathbf{CSMod} \hookrightarrow {}_R \mathbf{SMod}$ (Hint: Prove that for every left R-semimodule M and every cancellative left R-semimodule N, there exists a natural isomorphism of commutative monoids $Hom_R(c(M), N) \simeq Hom_R(M, N)$).

Q8. (20 points) Prove or disprove:

(a) Let R be a semiring. Then $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ is a short exact sequence of left R-semimodules if and only if $X \simeq Ker(g)$ and $Z \simeq Coker(f)$.

(b) For any semiring R, there is at least one morphism of semirings $f : R \longrightarrow \mathbb{B}$ (i.e. $char(R) \neq \varnothing$).

(c) If $f : M \longrightarrow N$ is a surjective morphism of left R-semimodules, then $M/Ker(f) \simeq N$.

Q9. (20 points) Indicate if each of the following statements is TRUE or FALSE (where R is a semiring):

(a) For every left R-semimodule M and every R-subsemimodule $L \leq_R M$, we have a short exact sequence of left R-semimodules

$$0 \longrightarrow L \xrightarrow{\iota} M \xrightarrow{\pi} M/L \longrightarrow 0,$$

where ι is the canonical injection and π is the canonical projection.

(b) $\mathbb{B} \oplus \mathbb{W}$ is an injective commutative monoid.

(c) $H := [0, 1)$ is an large ideal of $R = ([0, 1], \max, \min)$ (both R and I are considered as R-semimodules).

(d) If the semiring R is additively idempotent, then every left R-semimodule has an injective hull.

(e) If L, N are left R-semimodules with $L \cap N = \{0\}$, then $L + N = L \oplus N$ (a direct sum).

(f) If M is a left R-semimodule with two injective hulls E_1, E_2, then $E_1 \simeq E_2$.

(g) For every left R-semimodule M, we have two left exact functors

$$Hom_R(M, -) :_R \mathbf{SMod} \longrightarrow \mathbf{CMon} \text{ and } Hom_R(-, M) :_R \mathbf{SMod} \longrightarrow \mathbf{CMon}.$$

(h) Every ideal-simple left R-semimodule is congruence-simple.

(i) Every congruence-simple left R-semimodule is ideal-simple.

(j) $\mathbb{W}$ contains a subsemiring isomorphic to $\mathbb{Q} \cap [0, \infty)$.

Bonus: Define a commutative semiring as a datum $(S, +, \cdot)$ in which $(S, +)$ and $(S, \cdot)$ are commutative semigroups such that $(\cdot)$ distributes over $(+)$ from both sides. Find all congruence-simple commutative semirings of order 2 (up to isomorphism).

GOOD LUCK