King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 655: Applied & Computational Algebra

Final Exam, Spring Semester 242 (180 minutes)

Prof. Jawad Abuihlail

==

Remark: Solve **7** questions including **Q8 and Q9.** Show **full details.**

Throughout, and *unless otherwise explicitly mentioned*, R is a commutative ring with $1_R \neq 0_R$ with total quotient ring $T(R)$. With $\Gamma(R)$ we denote the *zero-divisor graph* of R and with $diam(\Gamma(R))$ (resp. $gr(\Gamma(R))$) its diameter (resp. girth).

Q1. (14 points) Show that

(a) If $a - b - c - d$ is a *minimal path* from a to d in $\Gamma(R)$, then $a, d \in Z(R)^* \backslash Nil(R)$.

(b) If $\Gamma(R)$ has partitions X and Y with cut-vertex a. then $\{0, a\}$ is an ideal of R.

Q2. (14 points) Show that

(a) $\chi(\Gamma_0(R)) = 2$ if and only if R is an integral domain, $R \simeq \mathbb{Z}_4$ or $R \simeq \mathbb{Z}_2[x]/(x^2)$.

(b) If R is a coloring (i.e. $\chi(\Gamma_0(R)) < \infty$) and P is an associated prime, then P is a maximal ideal or R_P is a field. (**Hint:** Use the fact that if x is a nilpotent element and $|Rx| = \infty$, then R contains an infinite clique).

Q3. (14 points) Let R be a finite commutative ring with $Z(R) \neq 0$ an ideal. Show that

(a) $Z(R) = Nil(R)$.

(b) There exists some $b \in Z(R)^*$ such that $bZ(R) = \{0\}$ (hence $\Gamma(R)$ is star-shaped reducible).

Q4. (14 points) Show that

(a) If $\Gamma(R)$ contains a vertex a with $\deg(a) \geq 3$, then $\Gamma(R \times \mathbb{Z}_3)$ is not planar.

(b) $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is not planar.

Q5. (14 points) Let I be a proper ideal of R. Show that

(a) $\Gamma_I(R) = \Gamma(R)$ if and only if $I = \{0\}$ or R is an integral domain with I a *prime* ideal (*i.e.* both graphs are the order-zero graph).

(b) Consider $R = \mathbb{Z}_4 \times \mathbb{Z}_2$, $I := \{0\} \times \mathbb{Z}_2$ and $J := \{0, 2\} \times \mathbb{Z}_2$. Show that $\Gamma_I(R)$ is a induced subgraph of $\Gamma(R)$ while $\Gamma_J(R)$ is not a subgraph of $\Gamma(R)$.

Q6. (14 points) Let S be a commutative semigroup with absorbing 0 and $Z(S)$ the ideal of zero-divisors. Show that

(a) If $a - x - b$ is a path in $\Gamma(S)$, then either $\{0, x\}$ is an ideal of S *or* $a - x - b$ is contained in a cycle of length at most 4.

(b) If $|Z(S)| \geq 3$ and $\{0, x\}$ is *not* an ideal of S for any $x \in S^*$, then any edge $a - x$ in $\Gamma(S)$ is contained in a cycle of length at most 4 (therefore $\Gamma(S)$ is a union of triangles and squares).

Q7. (14 points) Let I be proper ideals of a semiring S. Show that

(a) I is a primary ideal of S if and only if $V(\Gamma_I(S)) \cup I$ is a prime ideal of S.

(b) Let J be a proper ideal of S. Then $\Gamma_I(S) = \Gamma_J(S)$ if and only if $I = J$.

Q8. (12 points) State whether each of the following statement is TRUE of FALSE:

1. $\chi(\Gamma_0(\mathbb{Z}_8)) = 3$.
2. $\Gamma(\mathbb{Z}_{12})$ has no cut-vertex.
3. $\Gamma(\mathbb{Z}_{12})$ is Planar.
4. $\Gamma(\mathbb{Z}_{243})$ is star-shaped reducible.
5. If S is a commutative semigroup with absorbing 0, then $\Gamma(S)$ is connected with $diam(\Gamma(S)) \leq 3$.
6. There is a commutative ring R such that $\Gamma(R)$ is a hexagon.
7. If $\Gamma(R)$ is complete, then $Z(R) = Nil(R)$.
8. For every prime p, there exists a commutative ring R with $\Gamma(R) = K_{p-1}$.
9. If R is finite and $\Gamma(R)$ is a regular graph, then it is either a complete graph or a complete bipartite graph.
10. There is a commutative ring R with no identity such that $gr(\Gamma(R)) \notin \{3, 4, \infty\}$.
11. $diam(\Gamma(T(R))) = diam(\Gamma(R))$ and $gr(\Gamma(T(R))) = gr(\Gamma(R))$.
12. For any commutative rings R_1 and R_2, $diam(R_1 \times R_2) = 3$.

Q9. (18 points) Prove or disprove:

(a) For any proper ideal I of R, there exists a commutative ring S such that $\Gamma_I(R) \simeq \Gamma(S)$.

(b) There is a commutative semigroup S with absorbing 0 such that $\Gamma(S)$ is a pentagon $a - b - c - d - e - a$.

(c) Given $n \geq 3$, there is a commutative ring R_n with $\Gamma(R_n)$ finite and containing a cycle of length n.

GOOD LUCK