King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 655: Applied & Computational Algebra

Midterm Exam, Spring Semester 242 (120 minutes)

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Remark: Solve **6** questions including **Q8.** Show **full details.**

Throughout, and *unless otherwise explicitly mentioned*, R is a commutative ring with $1_R \neq 0_R$. With $\Gamma(R)$ we denote the *zero-divisor graph* of R and with $diam(\Gamma(R))$ (resp. $gr(\Gamma(R))$) its diameter (resp. girth).

Q1. (14 points) Show that
(a) $\Gamma(R)$ is finite if and only if R is finite or an integral domain.
(b) $\Gamma(R)$ is connected.

Q2. (14 points) Show that
(a) $gr(\Gamma(R)) \in \{3, 4, \infty\}$.
(b) Give examples of commutative rings R_1, R_2 and R_3, such that

$$gr(\Gamma(R_1)) = 3,\ gr(\Gamma(R_2)) = 4 \text{ and } gr(\Gamma(R_3)) = \infty.$$

Q3. (14 points) Show that
(a) Show that $diam(\Gamma(R)) \leq 3$.
(b) Give examples of commutative rings R_1, R_2, R_3 and R_4, such that

$$diam(\Gamma(R_1)) = 0,\ diam(\Gamma(R_2)) = 1,\ diam(\Gamma(R_3)) = 2 \text{ and } diam(\Gamma(R_3)) = 3.$$

Q4. (14 points) Recall that the idealization of an R-module M into R is $R(+)M := R \times M$ with pointwise addition an multiplication given by

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_2).$$

Assume that $M := \mathbb{Z}_3$ is an R-module. Show that
(a) If $ann_R(\mathbb{Z}_3) \neq 0$, then $gr(\Gamma(R(+)\mathbb{Z}_3)) = 3$.
(b) If $ann_R(\mathbb{Z}_3) = 0$, then $gr(\Gamma(R(+)\mathbb{Z}_3)) = \infty$ (show that in this case $R \simeq \mathbb{Z}_3$ as rings).

Q5. (14 points) Let S be a commutative semiring and I a non-zero ideal of S. Define $\Gamma_I(S)$ to be the graph with vertices

$$Z_I(S) := \{x \in S \backslash I \mid \ xy \in I \text{ for some } y \in S \backslash I\}$$

and two distinct vertices x and y are adjacent iff $xy \in I$. Show that

(a) $\Gamma_I(S)$ is connected.

(b) $diam(\Gamma_I(S)) \leq 3$.

(**Hint:** If x, y are non-adjacent vertices, then consider the different cases for x^2, y^2 to be in I or $S \backslash I$).

Q6. (14 points) Show that

(a) $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is complete, but $\Gamma((\mathbb{Z}_2 \times \mathbb{Z}_2)[x])$ is not complete.

(b) If $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then

$$\Gamma(R) \text{ is complete } \Longleftrightarrow \Gamma(R[x]) \text{ is complete } \Longleftrightarrow \Gamma(R[[x]]) \text{ is complete.}$$

(**Hint:** If $f(x) = \sum\limits_{i=0}^{\infty} f_i x^i \in R[[x]]$ and for some nonnegative integer t, $f_t \notin Z(R)$, while $f_i \in N(R)$ for $0 \leq i \leq t-1$, then $f(x) \notin Z(R[[x]])$).

Q7. (14 points) Let $V := \mathbb{R}^{(\mathbb{N})}$ and consider the non-commutative ring $R := End_{\mathbb{R}}(V)$ under point-wise addition and multiplication taken to be the composition of functions, so

$$Z(R) = \{f \in R \mid f \circ g = 0 \text{ or } g \circ f = 0 \text{ for some } g \in R^*\}.$$

Consider the non-empty directed graph $\overrightarrow{\Gamma}(R)$ in which the set of vertices is $Z(R)^*$ and for two distinct elements $f, g \in Z(R)^*$ there is an arrow $f \longrightarrow g$ iff $f \circ g = 0$. Show that

(a) $\overrightarrow{\Gamma}(R)$ is disconnected.

(b) $\overrightarrow{\Gamma}(R)$ contains an *infinite* complete subgraph.

Q8. (30 points) Prove or disprove:

(a) If $Z(R)$ is an ideal, then $Z(R)$ is prime.

(b) If R_1 and R_2 are commutative rings such that $\Gamma(R_1) \simeq \Gamma(R_2)$, then $R_1 \simeq R_2$.

(c) There is a commutative ring R with $\Gamma(R) \simeq P_4$.

GOOD LUCK