King Fahd University of Petroleum and Minerals
Department of Mathematics
Math 665 Final Exam
The Second Semester of 2022-2023 (222)
Time Allowed: 150mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		20
2		20
3		20
4		20
5		20
Total		100

Problem 1:

Consider a twice continuously differentiable function $f : [-\pi, \pi] \to \mathbb{R}$ such that

$$f(-\pi) = f(\pi) \quad \text{and} \quad f'(-\pi) = f'(\pi) \tag{1}$$

1.)a.)(8pts) Relying on the eigenvalue problem for the operator $\frac{d^2}{dx^2}$ subject to the boundary conditions (1), find functions $f_n : [-\pi, \pi] \to \mathbb{R}$ and real numbers A_n such that

$$f(x) = \sum_{n=0}^{\infty} A_n f_n(x), \quad \forall x \in [-\pi, \pi]$$

b.)(4pts) Compute A_n explicitly, for $f(x) = \cos x$. Justify your answers clearly.

2.)(8pts) Find functions $g_n : [-\pi, \pi] \to \mathbb{C}$ and complex numbers C_n such that

$$f(x) = \sum_{n=-\infty}^{\infty} C_n g_n(x), \quad \forall x \in [-\pi, \pi]$$

Solution:

1.) a.) W consider the eigenvalue problem $y''+\lambda y = 0$, with $y(-\pi) = y(\pi)$ & $y'(-\pi) = y'(\pi)$. The auxiliary equation is $m^2 + \lambda = 0$. Assume $\lambda = -\alpha^2$, $\alpha > 0$. Thus, $m^2 - \alpha^2 = 0$, and we deduce the solution $y(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$. The boundary conditions imply that $c_1 \cosh \alpha\pi - c_2 \sinh \alpha\pi = c_1 \cosh \alpha\pi + c_2 \sinh \alpha\pi$ and $-c_1\alpha \sinh \alpha\pi + c_2\alpha \cosh \alpha\pi = c_1\alpha \sinh \alpha\pi + c_2\alpha \cosh \alpha\pi$, that is, $c_1 = c_2 = 0$. Now, assume that $\lambda = 0$. This implies that $y = c_1 x + c_2$, and the boundary conditions imply that $-c_1\pi + c_2 = c_1\pi + c_2$, that is, $c_1 = 0$, and we get the constant solution $y_0(x) = 1$. Lastly, we assume Assume $\lambda = \alpha^2$, $\alpha > 0$.. Thus, $m^2 + \alpha^2 = 0$, and we deduce the solution $y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$. The boundary conditions imply that $c_1 \cos \alpha\pi - c_2 \sin \alpha\pi = c_1 \cos \alpha\pi + c_2 \sin \alpha\pi$ and $-c_1\alpha \sin \alpha\pi + c_2\alpha \cos \alpha\pi = c_1\alpha \sin \alpha\pi + c_2\alpha \cos \alpha\pi$, that is, $\sin \alpha\pi = 0$ if $c_2 \neq 0$ and/or $c_1 \neq 0$. We then get the solution $y_n(x) = \cos(nx)$ and $\tilde{y}_n(x) = \sin(nx)$, for $n = 1, 2$. The functions $\{y_0, y_n\}$ consists of a complete orthogonal basis of the Hilbert space $H = \{\varphi \in \mathcal{C}^2([-\pi, \pi]),\ \varphi(-\pi) = \varphi(\pi) \text{ and } \varphi'(-\pi) = \varphi'(\pi)\}$. Thus,

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx), \ \ x \in [-\pi, \pi]. \tag{2}$$

And, we have $A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$, and $B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$.

b.) We have $A_0 = \frac{1}{2\pi} \underbrace{\int_{-\pi}^{\pi} \cos x dx}_{=[\sin x]_{-\pi}^{\pi}]=0}$, $A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x \cos(nx) dx$, and $B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x \sin(nx) dx$.

We have $\cos x \cos(nx) = \frac{1}{2}[\cos(1+n)x + \cos(n-1)x]$. This imply $A_n = 0$, $n \neq 1$ & $A_1 = 1$.
We have $\cos x \sin(nx) = \frac{1}{2}[\sin(1+n)x + \sin(n-1)x]$. This imply $B_n = 0$, $n \neq 1$ & $B_1 = 0$.

2.) We notice that the functions $\{y_n + i\tilde{y}_n(x)\}$, $n = 0, \pm 1, \pm 2, ...$, consists of a complex complete orthogonal basis of H, and we

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}, \ \ x \in [-\pi, \pi]. \tag{3}$$

Finally, we have $C_n = \frac{\int_{-\pi}^{\pi} f(x) e^{inx} dx}{\int_{-\pi}^{\pi} |e^{inx}|^2 dx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} dx$.

Problem 2:

1.)(10pts) Use separation of variables method to find the solution $u = u(r, \theta, t)$ of the BVP

$$\begin{cases} u_{rr} + \dfrac{1}{r}u_r + \dfrac{1}{r^2}u_{\theta\theta} = u_t, \\ u(2, \theta, t) = 0, \\ u(r, 0, t) = 0, \ u(r, \pi, t) = 0, \end{cases} \tag{4}$$

where $r \in (0, 2]$, $\theta \in [0, \pi]$ and $t \geq 0$.

2.) a.)(6pts)Find eigenvalues and eigenfunctions of the BVP

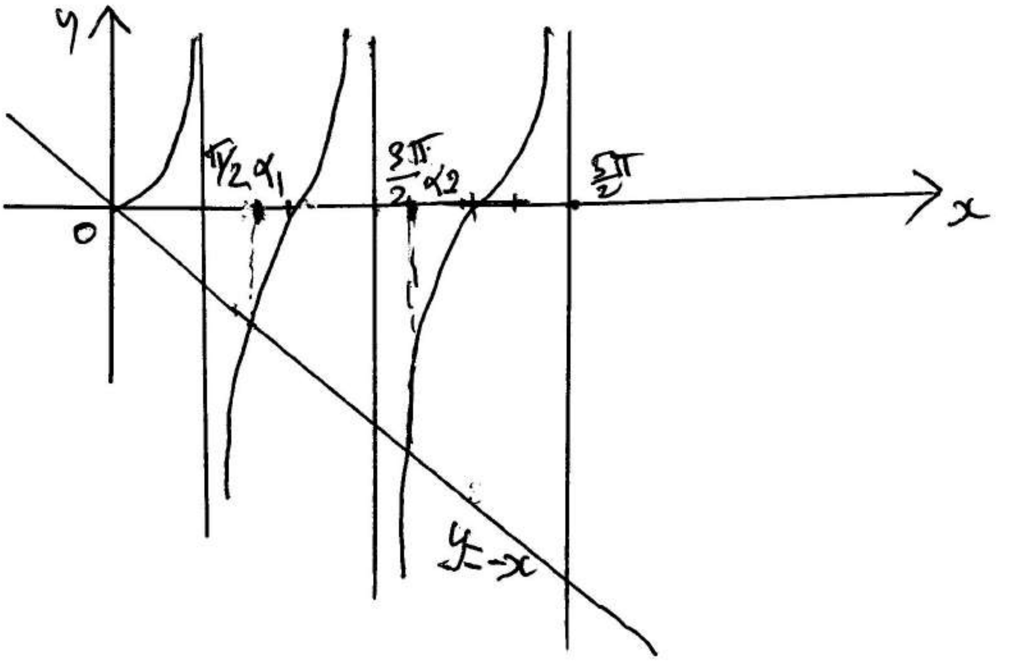
$$y'' + \lambda y = 0, \quad y(0) = 0, \ y(1) + y'(1) = 0.$$

b.)(4pts)Represent the first two eigenvalues on the real line.

__Solution:__

1.) We look for a solution in the form $u = RT\Theta$. Thus, $R''T\Theta + \frac{1}{r}R'T\Theta + \frac{1}{r^2}RT\Theta'' = RT'\Theta$, so that $\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \frac{T'}{T}$. We uncoupled as follows $\frac{T'}{T} = -\lambda$, $\frac{\Theta''}{\Theta} = -\mu$, and $\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\mu = -\lambda$. From the boundary conditions we deduce that $R(2) = 0$, $\Theta(0) = 0$ and $\Theta(\pi) = 0$. So, we consider the $T' + \lambda T = 0$ and the problem problems $\begin{cases} \Theta'' + \mu\Theta = 0 \\ \Theta(0) = \Theta(\pi) = 0 \end{cases}$ and $\begin{cases} r^2R'' + rR' + (\lambda r^2 - \mu)R = 0 \\ R(2) = 0 \end{cases}$. We know that the problem in Θ has solutions $\Theta_m(\theta) = \sin(m\theta)$ only for $\mu = m^2$, $m = 1, 2, \ldots$ Next, the equation for R is the Bessel eigenvalue problem. Its solutions are $R_m(r) = J_m(\sqrt{\lambda}r)$, where $J_m(r)$ is the Bessel functions of order m. We must have $J_m(2\sqrt{\lambda}) = 0$. Let α_n such that $J_m(\alpha_n) = 0$, $n = 1, 2, ..$, so that $2\sqrt{\lambda_n} = \alpha_n$, that is, $\lambda_n = \frac{1}{4}(\alpha_n)^2$. Finally, the equation for T has the solution $T_n = e^{-\lambda_n t}$ Thus, $u(r, \theta, t) = \sum_{n,m=1}^{\infty} A_{nm}J_m(\sqrt{\lambda_n}r)\sin(m\theta)e^{-\lambda_n t}$.

2.) We refer the reader to the solution of Problem 1. Assume $\lambda = -\alpha^2$, $\alpha > 0$. We have $y(x) = c_1\cosh(\alpha x) + c_2\sinh(\alpha x)$. The boundary conditions imply that $c_1 = 0$ and $c_2 = 0$ as $\alpha\cosh\alpha + \sinh\alpha > 0$. Now, assume that $\lambda = 0$. Thus $y(x) = c_1x + c_2$, and the boundary conditions imply that $c_2 = 0$, and also $2c_1 = 0$, that is $c_1 = 0$. Lastly, we assume Assume $\lambda = \alpha^2$, $\alpha > 0$. Thus, $y(x) = c_1\cos(\alpha x) + c_2\sin(\alpha x)$. The boundary conditions imply that $c_1 = 0$ and and $\alpha\sin\alpha + \cos\alpha = 0$ (if $c_2 \neq 0$). Notice that $\alpha = n\pi$ and $\frac{\pi}{2} + n\pi$ are not solutions of this equation. Thus, α_n satisfies $\tan\alpha_n = -\alpha_n$. Eigenvalues are $\lambda_n = \alpha_n^2$ and the corresponding eigenvalues are $y_n(x) = \sin(\alpha_n x)$, $n = 1, 2, \ldots,$.



Problem 3:

1.)(10pts) Knowing that $y_1(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ is one power series solution of the DE

$$xy'' + (1-x)y' - y = 0, \tag{5}$$

use the formula $y_2(x) = y_1(x) \int y_1^{-2}(x) e^{-\int P(x)dx} dx$, where the DE is rewritten in the form $y'' + P(x)y' + Q(x) = 0$, to find a second power series solution y_2 of the DE (5) that is linearly independent to y_1.

2.)(10pts) Consider the BVP

$$\begin{cases} u'' - u = x, \\ u(0) - u'(0) = \alpha, \\ u(1) - u'(1) = \beta. \end{cases} \tag{6}$$

Use the Fredholm alternative theorem to find conditions on α and β that guarantee the existence of solution to the BVP.

Solution:

1.) First, we find $P(x) = \frac{1}{x} - 1$, $\int P(x)dx = \ln x - x$, and then $e^{-\int P(x)dx} = \frac{e^x}{x}$. Thus, $y_2(x) = y_1(x) \int e^{-2x} \frac{e^x}{x} dx = y_1(x) \int \frac{e^{-x}}{x} dx$. Now, we substitute e^{-x} as power series, and we find $y_2(x) = y_1(x) \int \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} dx = y_1(x) \int \frac{1}{x}(1 + \sum_{n=1}^{\infty} \frac{(-x)^n}{n!}) dx$. Lastly, we notice that $\sum_{n=1}^{\infty} \frac{(-x)^n}{n!} = -x \sum_{n=1}^{\infty} \frac{(-x)^{n-1}}{n!}$, so that $y_2(x) = y_1(x)[\ln x - \sum_{n=1}^{\infty} \int \frac{(-x)^{n-1}}{n!} dx]$, and $y_2(x) = e^x[\ln x + \sum_{n=1}^{\infty} \frac{(-x)^n}{nn!}]$.

2.) a.) Let the operator $L = \frac{d^2}{dx^2} + 1$. We have $\int_0^1 vLudx = \int_0^1 (vu'' + uv)dx = [u'v - uv']_0^1 + \int_0^1 uLvdx$. The the Fredholm alternative theorem says that, the equation $Lu = x$ has a solution if $\int_0^1 xvdx = [u'v - uv']_0^1$, for all v that satisfies the problem

$$\begin{cases} v'' - v = 0, \\ v(0) - v'(0) = 0, \\ v(1) - v'(1) = 0. \end{cases} \tag{7}$$

The general solution is this equation is $v(x) = c_1 e^x + c_2 e^{-x}$. The boundary conditions imply that $c_2 = 0$. Thus $v(x) = ce^x$. Now, we require that $\underbrace{\int_0^1 xe^x dx}_{=[(x-1)e^x]_0^1 = 1} = \underbrace{[u'e^x - ue^x]_0^1}_{=e(u'(1)-u(1))-(u'(0)-u(0))}$,

that is, $1 = -e\beta + \alpha$.

Problem 4:

1.) a.)(3pts) Given that $y_1(x) = \cos(2\pi \ln x)$ and $y_2(x) = \sin(2\pi \ln x)$ are two solutions of DE $x^2y'' + xy' + 4\pi^2 y = 0$ on $(\frac{1}{3}, \infty)$. only check that the two solutions are linearly independent [I am not asking you to solve this DE].

b.)(4pts) Represent the zeros of the two functions y_1 and y_2 in the interval $[e^{-\frac{1}{2}}, e^{\frac{1}{2}}]$ [I am not asking to graph the functions y_1 and y_2].

c.)(3pts) Does the Sturm separation theorem satisfied?

2.) Using the Sturm comparison theorem and its Corollary, what can you say about the distance between two consecutive zeros of solutions of the DE $y'' + (9 - \frac{\gamma}{x^2})y = 0$, for

a.)(4pts) $\gamma < 0$;

b.)(3pts) $\gamma = 0$;

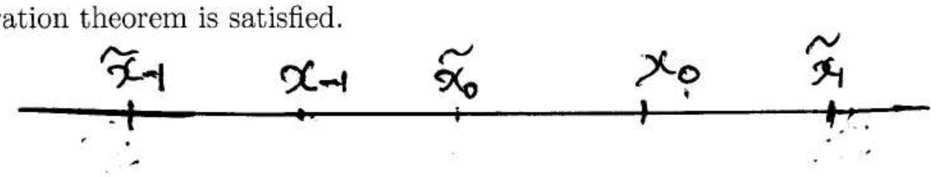
c.)(3pts) $\gamma > 0$.

Solution:

1.)a.) The Wronskian of y_1 and y_2 is $W = \begin{vmatrix} \cos(2\pi \ln x) & \sin(2\pi \ln x) \\ -\frac{2\pi}{x}\sin(2\pi \ln x) & \frac{2\pi}{x}\cos(2\pi \ln x) \end{vmatrix} = \frac{2\pi}{x} \neq 0.$

b.) The zeros of y satisfy $\cos(2\pi \ln x_n) = 0$, that is, $2\pi \ln x_n = \frac{\pi}{2} + n\pi$, $n = 0, \pm 1, \pm 2, ...$, and then $x_n = e^{\frac{1}{4}+\frac{n}{2}}$, so we have $x_0 = e^{\frac{1}{4}}$ and $x_{-1} = e^{-\frac{1}{4}}$.

Similarly, the zeros of y_2 satisfy $\sin(2\pi \ln \tilde{x}_n) = 0$, that is, $2\pi \ln \tilde{x}_n = n\pi$, $n = 0, \pm 1, \pm 2, ...$, and then $\tilde{x}_n = e^{\frac{n}{2}}$, so we have $\tilde{x}_0 = 1$, $\tilde{x}_1 = e^{\frac{1}{2}}$ and $\tilde{x}_{-1} = e^{-\frac{1}{2}}$. c.) Yes, between two consecutive zeros of of one solution, there is exactly on zero of the other solution. Thus, the Sturm separation theorem is satisfied.



2.) Consider the equation

$$y'' + (9 - \frac{\gamma}{x^2})y = 0. \tag{8}$$

Let two consecutive zeros α and β of a solution y of (8).

a.) Now, let us consider the solution $z(x) = \sin 3(x - \alpha)$ of the DE $z'' + 9z = 0$. The zeros of z are $x = \alpha + \frac{n\pi}{3}$, $n = 0, \pm 1, \pm 2, ...,$.Since $9 - \frac{\gamma}{x^2} > 9$ and $y(\alpha) = z(\alpha) = 0$, the comparison theorem says that $\beta \in (\alpha, \alpha + \frac{\pi}{3})$, that is, $\beta - \alpha < \frac{\pi}{3}$.

b.) <u>Method 1:</u>

A corollary of the comparison theorem says that if $m \le q(x) \le M$, then $\frac{\pi}{\sqrt{M}} \le \beta - \alpha \le \frac{\pi}{\sqrt{m}}$. Here we can set $q(x) = m = M = 9$. It follows that $\beta - \alpha = \frac{\pi}{3}$.

<u>Method 2:</u>

We apply the comparison theorem of the same equation $y'' + 9y = 0$ and $z'' + 9z = 0$. A $9 \ge 9$, we get exactly like in part (b) that $\beta - \alpha \le \frac{\pi}{3}$. Also, between α and β there is a zero of $z(x) = \sin(3(x - \alpha))$, that is, $\alpha + \frac{\pi}{3}$. This means that $\beta - \alpha \ge \frac{\pi}{3}$. In conclusion, it follows that $\beta - \alpha = \frac{\pi}{3}$.

c.) as $9 > 9 - \frac{\gamma}{x^2}$, the comparison theorem says that $\alpha + \frac{\pi}{3} \in (\alpha, \beta)$, since $z(\alpha) = 0$. This means that $\beta - \alpha > \frac{\pi}{3}$.

Problem 5:

Say if the DE is oscillatory or non-oscillatory. Justify your answer.
1.)(10pts) $u'' + \frac{1}{x(x+1)}u' = 0$ on $(0, \infty)$, $u \neq$ constant.

2.)(10pts) $y'' + \frac{1}{x+1}y' + (\frac{1}{4x^2} + y^2)y = 0$ on $(0, \infty)$. You can consider writing the differential equation satisfied by the function $v = y\sqrt{x+1}$.

<u>Solution:</u>
1.) this equation is not in the form $y'' + q(x)y = 0$, and we cannot apply the oscillation theorem to it. We instead solve it to see if we can find a non-oscillatory solution, in this equation the equation is non-oscillatory.
Let set $w = u'$. We reduce the DE into a first order DE $w' + \frac{1}{x(x+1)}w = 0$, that is, $\int \frac{dw}{w} = -\int \frac{1}{x(x+1)}dx = \int (\frac{1}{(x+1)} - \frac{1}{x})dx$, and $\ln|w| = \ln|\frac{x+1}{x}| + c$, and $w(x) = c\frac{x+1}{x}$. We then deduce that $u(x) = c\int \frac{x+1}{x}dx = c\int (1 + \frac{1}{x})dx = c(x + \ln x) + c_2$. We just found two non-oscillatory solutions $u(x) = 1$ and $u(x) = x + \ln x$, this equation is non-oscillatory.

2.) Let $y = (x+1)^{-\frac{1}{2}}v$. So, $y' = -\frac{1}{2}(x+1)^{-\frac{3}{2}}v + (x+1)^{-\frac{1}{2}}v'$ and
$y'' = \frac{3}{4}(x+1)^{-\frac{5}{2}}v - (x+1)^{-\frac{3}{2}}v' + (x+1)^{-\frac{1}{2}}v''$. We substitute into the DE to find after simplification

$$\frac{3}{4}(x+1)^{-2}v + v'' - \frac{1}{2}(x+1)^{-2}v + (\frac{1}{4x^2} + (x+1)^{-1}v^2)v = 0,$$

that is

$$v'' + \underbrace{(\frac{1}{4x^2} + \frac{1}{4(x+1)^2}}_{=q(x)} + \underbrace{(x+1)^{-1}v^2)}_{\geq 0}v = 0.$$

We can see that $\lim_{x\to+\infty} q(x) = \frac{1}{2} > \frac{1}{4}$. From an oscillation theorem, the equation for v is oscillatory, and so for the equation in y as both equations are equivalent on $(0, \infty)$.