**King Fahd University of Petroleum and Minerals**
**Department of Mathematics**
**Math 665 Midterm Exam**
**The Second Semester of 2022-2023 (222)**
**Time Allowed: 120mn**

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Name: ID number:

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Textbooks are not authorized in this exam

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| Problem # | Marks | Maximum Marks |
|---|---|---|
| 1 | | 20 |
| 2 | | 20 |
| 3 | | 20 |
| 4 | | 20 |
| 5 | | 20 |
| Total | | 100 |

**Problem 1:**
Consider the autonomous system

$$\frac{dx}{dt} = y - (x - x^3)$$
$$\frac{dy}{dt} = -x^3$$

1.) (10pts) Find a domain $D$ of $\mathbb{R}^2$ and a positive definite function $V(x, y)$ such that $\frac{dV}{dt}$ is negative on $D$.
2.)(10pts) Find the largest domain of asymptotic stability of the origin.

**Solution:**

1.) $\frac{dx}{dt} = y - (x - x^3) \quad \times y' \Rightarrow \quad x'y' = \frac{1}{2}\frac{d}{dt}y^2 + x^3(x - x^3)$

$\frac{dy}{dt} = -x^3 \quad \times -x' \Rightarrow \quad -x'y' = \frac{1}{4}\frac{d}{dt}x^4$

$$\frac{1}{2}\frac{d}{dt}\left(y^2 + \frac{1}{2}x^4\right) = -x^3(x - x^3)$$

Let $V(x,y) = y^2 + \frac{1}{2}x^4$. $V$ is positive definite on $\mathbb{R}^2$.

$V^*(x,y) = -2x^4(1 - x^2)$

| $x$ | | $-1$ | | $0$ | | $1$ | |
|---|---|---|---|---|---|---|---|
| $1-x^2$ | $-$ | $0$ | $+$ | | $+$ | $0$ | $-$ |

$\Rightarrow V^*$ is negative definite on $D = \{(x,y), -1 < x < 1, -\infty < y < \infty\}$

2.) $V^*(x,y) = 0 \Leftrightarrow x = 0$ or $x = 1$, or $x = -1$

$\Rightarrow E = \{(x,y) \in D / V^*(x,y) = 0\} = \{(0,0)\}$

The critical solutions of the system are

$\begin{cases} y - (x - x^3) = 0 \\ x^3 = 0 \end{cases} \Rightarrow x = 0, y = 0$

The only invariant set of $E$ is the origin.

$C_\lambda = \{(x,y) \in D, \ V(x,y) < \lambda\}$

The boundary of $D$ are $x = -1$ and $x = 1$

$C_\lambda$ meets the boundary $V(-1,0) = V(1,0) = \frac{1}{2}$

$\Rightarrow$ The largest domain of asymptotic stability of the origin is $C_{\frac{1}{2}}$.

**Problem 2:**

1.)(10pts) Show that the origin is stable for the non-autonomous system

$$\frac{dx}{dt} = y,$$
$$\frac{dy}{dt} = -(1+e^{-t})x,$$

Hint: evaluate $\frac{d}{dt}[(1+e^{-t})x^2]$ and use it to find a Lyapuvov function $V(x,y,t)$.

2.)(10pts) $V(X,t)$ is infinitesimal upper bound $\iff \forall \varepsilon, \exists \delta > 0$ such that $|V(X,t)| < \varepsilon$, $\forall (X,t) \in \{(X,t) \in \mathbb{R}^2 \times \mathbb{R}^+, \ \|X\| < \delta\}$.

Is the function $U(x,y,t) = \frac{t}{1+t}x^2 + \frac{1}{2+t}y^2$ infinitesimal upper bound or no?

**Solution:**

1.) $\frac{d}{dt}[(1+e^{-t})x^2] = -e^{-t}x^2 + 2(1+e^{-t})x\dot{x}$

$\dot{x} = y \qquad \times\ y' \quad \Rightarrow \quad \dot{x}\dot{y} = \frac{1}{2}\frac{d}{dt}y^2$

$\dot{y} = -(1+e^{-t})x \qquad \times(-x') \quad \Rightarrow \quad -\dot{x}\dot{y} = (1+e^{-t})x\dot{x} = \frac{1}{2}\frac{d}{dt}[(1+e^{-t})x^2] + \frac{e^{-t}x^2}{2}$

$$\frac{1}{2}\frac{d}{dt}[y^2 + (1+e^{-t})x^2] = -\frac{1}{2}e^{-t}x^2$$

Let $V(x,y,t) = y^2 + (1+e^{-t})x^2$, $V^*(x,y,t) = -e^{-t}x^2$

$V(x,y,t) \geq y^2 + x^2 \qquad V^*(x,y,t) \leq 0$

$V$ is positive definite and $V^*$ is negative on $\{(x,y,t) / (x,y) \in \mathbb{R}^2, t \geq 0\}$

$\Rightarrow$ The origin is stable

2.) We have $\frac{t}{1+t} \leq 1$ and $\frac{1}{2+t} \leq 1$, $\forall t \geq 0$

$|U(x,y,t)| \leq x^2 + y^2$

For any $\varepsilon > 0$, $|x|^2 + |y|^2 \leq \delta = \varepsilon \Rightarrow |U(x,y,t)| < \varepsilon$

Yes, $U$ is an infinitesimal upper bound function.

**Problem 3:**

1.)(10pts) Analyze the bifurcation phenomenon in the ODE

$$\frac{dy}{dx} = y(4-y) - \mu, \quad \text{where } \mu \in \mathbb{R} \text{ is a parameter}$$

2) (10pts) Draw the bifurcation diagram.

**Solution:**

1) Critical points : $y(4-y)-\mu=0$, $y^2-4y+\mu=0$

$$\Delta = 16-4\mu = 4(4-\mu)$$

- if $\mu<4$, then $y_1=2-\sqrt{4-\mu}$, $y_2=2+\sqrt{4-\mu}$
- if $\mu=4$, then $y_3=2$
- if $\mu>4$, there is no critical point

Possible bifurcation points. $f(y)=y(4-y)-\mu$

$\begin{cases} f'(y)=4-2y \\ f(y)=0 \end{cases}$ . $f'(y)=0 \Rightarrow y_3=2,\ \mu=4$

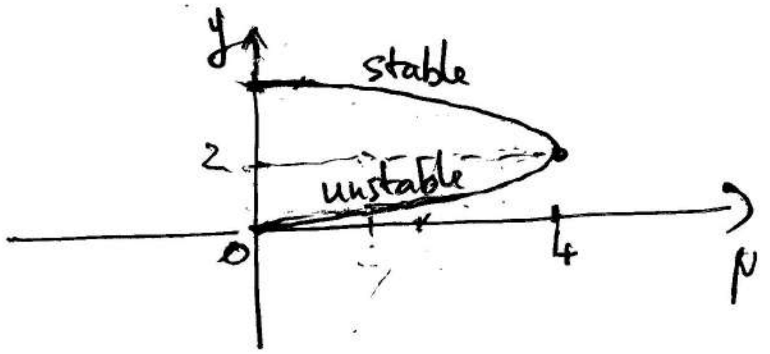
Stability of the critical points

Case 1: $\mu<4$: $f'(y_1)=2(2-y_1)=2\sqrt{4-\mu}>0 \Rightarrow y_1$ unstable

$f'(y_2)=2(2-y_2)=-\sqrt{4-\mu} \Rightarrow y_2$ stable

Case 2: $\mu=4$: $\dot{y}=-(y-2)^2$ $\xleftarrow{\quad}\underset{y=2}{+}\xleftarrow{\quad}$ $y_3$ semistable

2)




We trace the functions $f(\mu)=2-\sqrt{4-\mu}$ and $g(\mu)=2+\sqrt{4-\mu}$

**Problem 4:**

.)(10pts) What are the possible bifurcation points $(x_0, y_0, \mu_0)$ in the system

$$\frac{dx}{dt} = y - x$$
$$\frac{dy}{dt} = -y + \mu x - x^2, \quad \text{where } \mu \in \mathbb{R} \text{ is a parameter.}$$

2.)(10pts)Add the equation $\frac{d\mu}{dt} = 0$ to the system in part 1, and write the system in the form $X' = AX + F(X)$, where $X = (x, y, \mu)$ and $A$ is the Jacobian of the system at the origin. Compute $\lim \frac{\|F(X)\|}{\|X\|}$ as $\|X\| \to 0$.

**Solution:**

1.) Critical points are: $\begin{cases} y = x \\ -y + \mu x - x^2 = 0 \end{cases} \Rightarrow O\begin{pmatrix} 0 \\ 0 \end{pmatrix},\ A\begin{pmatrix} \mu-1 \\ \mu-1 \end{pmatrix}$

Bifurcation possibility. The Jacobian of the system at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

is $J = \begin{pmatrix} -1 & 1 \\ \mu - 2x_0 & -1 \end{pmatrix}$. Thus, $J\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ \mu & -1 \end{pmatrix}$, $J(A) = \begin{pmatrix} -1 & 1 \\ 2-\mu & -1 \end{pmatrix}$

Eigenvalues of the Jacobian?

At the origin $\begin{vmatrix} -1-\lambda & 1 \\ \mu & -1-\lambda \end{vmatrix} = 0$, $(1+\lambda)^2 = \mu \Rightarrow \begin{cases} \lambda = -1 \pm i\sqrt{-\mu}, & \mu < 0 \\ \lambda = -1, -1, & \mu = 0 \\ \lambda = -1 \pm \sqrt{\mu}, & \mu > 0 \end{cases}$

A Possible bifurcation at $\mu = 1$

At the point A. $\begin{vmatrix} -1-\lambda & 1 \\ 2-\mu & -1-\lambda \end{vmatrix} = 0$, $(1+\lambda)^2 = 2-\mu \Rightarrow \begin{cases} \lambda = -1 \pm i\sqrt{2-\mu}, & \mu > 2 \\ \lambda = -1, & \mu = 2 \\ \lambda = -1 + \sqrt{2-\mu}, & \mu < 2 \end{cases}$

A possible bifurcation at $\mu = 1$

Conclusion: There might be a bifurcation at the origin

2.) $J(0) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dot{X} = \underbrace{\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{J(0)} X + \underbrace{\begin{pmatrix} 0 \\ \mu x - x^2 \\ 0 \end{pmatrix}}_{F(X)}$

$X \neq 0 \Rightarrow \frac{|F(X)|}{|X|} = \frac{|\mu x - x^2|}{|x| + |y| + |\mu|} \leq \frac{|\mu||x| + |x|^2}{|x| + |y| + |\mu|} \leq \frac{|x|(|x| + |\mu| + |y|)}{|x| + |y| + |\mu|}$

$\frac{|F(X)|}{|X|} \leq |x|$, $\Rightarrow \lim\limits_{|X| \to 0} \frac{|F(X)|}{|X|} = 0$.

for $|X| \neq 0$

**Problem 5:**

1.)(10pts) Find eigenfunctions and eigenvalues of the the Sturm-Liouville BVP

$$y'' - y + \lambda y = 0,$$
$$y'(0) = y'(\pi) = 0.$$

2.)(10pts) Consider the nonlinear and nonhomogeneous BVP

$$-y'' + y^2 = 1, \quad (1)$$
$$y(0) = y(1) = 0. \quad (2)$$

The eigenfunctions and eigenvalues of the associated SL problem are $e_n(x) = \sin(n\pi x)$ and $\lambda_n = n^2\pi^2$. We assume $y^2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi x)$. Find the solution $y$ of the problem (1)-(2).

**Solution:**

1) The characteristic equation is $m^2 - 1 + \lambda = 0$.

- $\lambda - 1 = -\alpha^2$, $\alpha > 0$, $m^2 - \alpha^2 = 0$, $m = \pm\alpha \Rightarrow y = C_1 e^{\alpha x} + C_2 e^{-\alpha x}$, $y' = \alpha(C_1 e^{\alpha x} - C_2 e^{-\alpha x})$

  $\left.\begin{array}{l} y'(0) = 0 \Rightarrow C_1 = C_2 \\ y'(\pi) = 0 \Rightarrow C_1 e^{\pi\alpha} + C_2 e^{-\pi\alpha} = 0 \end{array}\right\} \Rightarrow C_1 = C_2 = 0$

- $\lambda = 1$, $m^2 = 0$, $m = 0, 0$, $y = C_1 x + C_2$, $y' = C_1$

  $y'(0) = y'(\pi) = 0 \Rightarrow C_1 = 0 \Rightarrow y = C_2$

- $\lambda - 1 = \alpha^2$, $\alpha > 0$, $m^2 + \alpha^2 = 0$, $m = \pm i\alpha$, $y = C_1 \cos\alpha x + C_2 \sin\alpha x$

  $y' = -C_1 \alpha \sin\alpha x + C_2 \alpha \cos\alpha x$, $y'(0) = 0 \Rightarrow C_2 = 0$

  $y'(\pi) = 0$, $\sin\alpha\pi = 0$, $\alpha\pi = n\pi$, $\alpha = n$

<u>Conclusion:</u> the eigenvectors are $e_n(x) = \cos nx$ and the eigenvalues are $\lambda_n = 1 + n^2$, $n = 0, 1, 2 \dots$

2.) $y = \sum_{n=1}^{\infty} C_n e_n$. We have $-\ddot{e}_n = \lambda_n e_n$

We substitute $y$ into the equation $-y'' + y^2 = 1$

$$\Rightarrow -\sum_{n=1}^{\infty} C_n \ddot{e}_n + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e_n = 1$$

$$\sum_{n=1}^{\infty} \left[C_n \lambda_n + \frac{(-1)^n}{n}\right] e_n = 1 \Rightarrow C_n \lambda_n + \frac{(-1)^n}{n} = \frac{\int_0^1 \sin(n\pi x)\,dx}{\int_0^1 \sin^2(n\pi x)\,dx}$$

$$\Rightarrow C_n \lambda_n + \frac{(-1)^n}{n} = \frac{2(1 - (-1)^n)}{n\pi} \Rightarrow C_n = \frac{1}{n^2\pi^2}\left[\frac{-(-1)^n}{n} + \frac{2(1 - (-1)^n)}{n\pi}\right]$$