King Fahd University of Petroleum and Minerals
Department of Mathematics
Math 667 **Final** Exam
The First Semester of 2021-2022 (211)

Time Allowed: 120mn

Name: _____   ID number: _____

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		20
2		20
3		20
4		20
5		20
Total		100

Problem 1: 1.) Write the following equations as a quasi-linear system

(5pt)

$$u^2 + \frac{x}{2}\frac{\partial u^2}{\partial x} + u\frac{\partial v}{\partial x} - vx\frac{\partial u}{\partial y} - \frac{1}{2}\frac{\partial v^2}{\partial y} = 0,$$

$$\frac{1}{2}\frac{\partial u^2}{\partial x} - u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial v^2}{\partial y} = 0.$$

(Do not solve it).

2.) Solve the quasilinear system   that solution passes through the

(15pt)   curve $\Gamma: u=-1, v=1$

$$u\frac{\partial}{\partial x}\begin{pmatrix}u\\v\end{pmatrix} - v\frac{\partial}{\partial y}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}\frac{u+1}{t+2}\\v^2\end{pmatrix}.$$ on $y=x$.

Solution:

1.) $u\left(u + x\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) - v\left(x\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = 0$

$u\frac{\partial}{\partial x}(xu+v) - v\frac{\partial}{\partial y}(xu+v) = 0$

and $u\frac{\partial}{\partial x}(u-v) - v\frac{\partial}{\partial y}(u-v) = 0$ $\Rightarrow$ $\boxed{u\frac{\partial}{\partial x}\begin{pmatrix}xu+v\\u-v\end{pmatrix} - v\frac{\partial}{\partial y}\begin{pmatrix}xu+v\\u-v\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}}$

2.) $\frac{dx}{dt} = u$; $\frac{dy}{dt} = -v$; $\frac{du}{dt} = \frac{u+1}{t+2}$; $\frac{dv}{dt} = v^2$

$\int\frac{du}{u+1} = \int\frac{dt}{t+2}$ $\int\frac{dv}{v^2} = \int dt$

$\ln|u+1| = \ln|t+2| + C$ $-\frac{1}{v} = t + C_2$

$\boxed{u+1 = C_1(t+2)}$ $\boxed{v = -\frac{1}{t+C_2}}$

Thus, $\frac{dx}{dt} = -1 + C_1(t+2)$

$\boxed{x = -t + C_1\frac{t^2}{2} + 2C_1 t + C_3}$

$\frac{dy}{dt} = \frac{1}{t+C_2} \Rightarrow y = \int\frac{dt}{t+C_2} = \ln|t+C_2| + C$

$\Rightarrow \boxed{e^y = C_4(t+C_2)}$

At $t=0$, $S = C_3$; $e^S = C_4 \cdot C_2$; $0 = 2C_1 \Rightarrow C_1 = 0$

$1 = -\frac{1}{C_2} \Rightarrow C_2 = -1$ • Thus, $C_4 = -e^S$

$\Rightarrow u = -1$; $v = -\frac{1}{t-1}$; $x = -t+S$; $e^y = -e^S(t-1)$

$$\boxed{u = -1 \text{ and } v e^y = e^{x+1-\frac{1}{v}}}$$

②

Problem 2: 1.) Consider the second order PDE

(5 pts)

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} = 1, \qquad (0.4)$$

Write this equation as a first order system of PDEs. **Do not solve it.**

2.) Solve the Cauchy problem

(15 pts)

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + 2\dfrac{\partial v}{\partial y} = 0, \\[2mm] \dfrac{\partial v}{\partial x} + 2\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial y} = 0, \\[2mm] u = X, \; v = 1 \text{ on } y = -x. \end{cases} \qquad (0.5)$$

Solution:

1) Let $V = \frac{\partial u}{\partial x}$ and $W = \frac{\partial u}{\partial y}$. We have $\frac{\partial W}{\partial x} = \frac{\partial V}{\partial y}$

$\Rightarrow \frac{\partial}{\partial x}V - \frac{\partial}{\partial y}W + \frac{\partial V}{\partial y} = 1$

$\Rightarrow \frac{\partial}{\partial x}\begin{pmatrix} V \\ W \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial y}\begin{pmatrix} V \\ W \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

2) $\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}_{A}\begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. A has two distincts eigenvalues $\lambda = -1, 3$

$(A+I)k_1 = 0 \Rightarrow k_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$; $(A-3I)k_2 = 0 \Rightarrow k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Thus, $\frac{\partial}{\partial x}\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \tilde{u}_y \\ \tilde{v}_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$(\Rightarrow) \begin{cases} \frac{\partial \tilde{u}}{\partial x} - \frac{\partial \tilde{u}}{\partial y} = 0 \quad \textcircled{1} \\ \tilde{u}(x, -x) = \frac{x-1}{2} \end{cases}$; $\begin{cases} \frac{\partial \tilde{v}}{\partial x} + 3\frac{\partial \tilde{v}}{\partial y} = 0 \quad \textcircled{2} \\ \tilde{v}(x, -x) = \frac{x+1}{2} \end{cases}$

P①: $\frac{dx}{dt} = 1, \frac{dy}{dt} = -1 \Rightarrow y = -x$ is a characteristic, $\frac{d\tilde{u}}{dt} = 0$

P② $\frac{dx}{dt} = 1, \frac{dy}{dx} = 3 \Rightarrow y = 3x$ is a characteristic, $\frac{d\tilde{v}}{dt} = 0$

On $y = -x, \frac{\partial}{\partial x} = -\frac{\partial}{\partial y} \Rightarrow \begin{cases} \frac{\partial \tilde{u}}{\partial x} = 0 \Rightarrow \tilde{u}(x, -x) = C \\ \frac{\partial \tilde{v}}{\partial x} = 0 \Rightarrow \tilde{v}(x, -x) = C \end{cases}$

This is incompatible with the initial conditions

Thus, the Cauchy problem has no solution. $\textcircled{3}$

Problem 3: Solve the Cauchy problem

(20pts)

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} - 4\dfrac{\partial v}{\partial y} = x, \\[2mm] \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} - 3\dfrac{\partial v}{\partial y} = 0, \\[2mm] u = 1, \ v = 0 \text{ on } y = x. \end{cases}$$

(0.6)

Solution:

$$\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

A has a repeated eigenvalue $\lambda = -1, -1$.

$$(A+I)K_1 = 0, \quad K_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}; \quad (A+I)K_2 = K_1, \quad K_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad T = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}; \quad T^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

$$\Rightarrow \frac{\partial}{\partial x}\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \tilde{u}_y \\ \tilde{v}_y \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}; \quad \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = T^{-1}\begin{pmatrix} u \\ v \end{pmatrix}; \quad \begin{pmatrix} \tilde{u}(x,x) \\ \tilde{v}(x,x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We solve the two following problems

$$(1) \begin{cases} \dfrac{\partial \tilde{u}}{\partial x} - \dfrac{\partial \tilde{u}}{\partial y} + \dfrac{\partial \tilde{v}}{\partial y} = 0 \\[2mm] \tilde{u}(x,x) = 0 \end{cases} \qquad (2) \begin{cases} \dfrac{\partial \tilde{v}}{\partial x} - \dfrac{\partial \tilde{v}}{\partial y} = x \\[2mm] \tilde{v}(x,x) = 1 \end{cases}$$

First, we solve ②.

$$\frac{dx}{dt} = 1; \quad \frac{dy}{dt} = -1; \quad \frac{d\tilde{v}}{dt} = x \Rightarrow x = t + c_1; \ y = -t + c_2; \quad \tilde{v} = \frac{t^2}{2} + c_1 t + c_3$$

At $t=0$, $s = c_1$, $s = c_2$; $1 = c_3 \Rightarrow x = t + s; \ y = -t + s \Rightarrow s = \dfrac{x+y}{2}$

$$\tilde{v} = \frac{t^2}{2} + st + 1 \qquad t = \frac{x-y}{2}$$

$$\Rightarrow \tilde{v}(x,y) = \frac{1}{2}(x^2 - xy + 2).$$

Next, we solve ①

$$\frac{dx}{dt} = 1; \quad \frac{dy}{dt} = -1, \quad \frac{d\tilde{u}}{dt} = -\frac{\partial \tilde{v}}{\partial y} = \frac{x}{2}.$$

$$\Rightarrow x = t + c_1; \ y = -t + c_2, \quad \tilde{u} = \frac{t^2}{4} + \frac{c_1}{2}t + c_3$$

At $t=0$, $s = c_1$; $s = c_2$, $0 = c_3 \Rightarrow \tilde{u} = \frac{t^2}{4} + \frac{s}{2}t; \begin{vmatrix} x = t + s \\ y = -t + s \end{vmatrix}$

$$\Rightarrow \tilde{u}(x,y) = \frac{1}{16}(3x^2 - y^2 - 2xy)$$

Finally,

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = T\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} 2\tilde{u} + \tilde{v} \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} \frac{1}{8}(7x^2 - 6xy - y^2 + 8) \\ \frac{1}{16}(3x^2 - y^2 - 2xy) \end{pmatrix}$$

④

Problem 4: Covert the following problem into solving a Laplace problem

(20pts)

$$\begin{cases} \dfrac{\partial u}{\partial x} + 4\dfrac{\partial u}{\partial y} + 2\dfrac{\partial v}{\partial y} = x, \\[2mm] \dfrac{\partial v}{\partial x} - 5\dfrac{\partial u}{\partial y} + 6\dfrac{\partial v}{\partial y} = y, \end{cases}$$

(0.7)

(**do not solve the obtained Laplace problem**).

Solution:

$$\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ -5 & 6 \end{pmatrix}\begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}. \qquad A \text{ has complex eigenvalues}$$

$$\underbrace{\qquad\qquad}_{A} \qquad\qquad \lambda = \ \pm 3$$

$$\text{Let} \begin{cases} x = \frac{1}{3}\eta \\ y = \xi + \frac{5}{3}\eta \end{cases} \Rightarrow \begin{cases} \xi = -5x + y \\ \eta = 3x \end{cases} \Rightarrow \begin{cases} \partial_x = -5\partial_\xi + 3\partial_\eta \\ \partial_y = \partial_\xi \end{cases}$$

$$\Rightarrow \quad -5u_\xi + 3u_\eta + 4u_\xi + 2v_\xi = \frac{\eta}{3} \qquad \Rightarrow -u_\xi + 3u_\eta + 2v_\xi = \frac{\eta}{3}$$

$$-5v_\xi + 3v_\eta - 5u_\xi + 6v_\xi = \xi + \frac{5}{3}\eta \Rightarrow v_\xi + 3v_\eta - 5u_\xi = \xi + \frac{5}{3}\eta$$

$$\Rightarrow \begin{cases} u_\eta - \frac{1}{3}u_\xi + \frac{2}{3}v_\xi = \frac{\eta}{9} \\[2mm] v_\eta - \frac{5}{3}u_\xi + \frac{1}{3}v_\xi = \frac{\xi}{3} + \frac{\xi}{9}\eta \end{cases} \Rightarrow \begin{pmatrix} u_\eta \\ v_\eta \end{pmatrix} + \underbrace{\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{5}{3} & \frac{1}{3} \end{pmatrix}}_{B}\begin{pmatrix} u_\xi \\ v_\xi \end{pmatrix} = \begin{pmatrix} \eta/9 \\ \frac{\xi}{3} + \frac{5}{9}\eta \end{pmatrix}$$

B has the complex eigenvalues $\lambda = \pm i$

$$(A - iI)K = 0 \Rightarrow K\begin{pmatrix} 1 \\ \frac{1}{2} + \frac{2}{2}i \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} + i\begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix}; \quad T = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

$$\text{Thus,} \begin{pmatrix} \tilde{u}_\eta \\ \tilde{v}_\eta \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \tilde{u}_\xi \\ \tilde{v}_\xi \end{pmatrix} = T^{-1}\begin{pmatrix} \eta/9 \\ \frac{\xi}{3} + \frac{5}{9}\eta \end{pmatrix} = \begin{pmatrix} \eta/9 \\ \frac{2}{9}\xi + \frac{\eta}{3} \end{pmatrix}$$

$$\Rightarrow \quad \tilde{u}_\eta + \tilde{v}_\xi = \eta/9 \qquad \Rightarrow \tilde{u}_{\eta\eta} + \tilde{v}_{\xi\eta} = \frac{1}{9} \quad \Rightarrow \tilde{u}_{\eta\eta} + \tilde{u}_{\xi\xi} = \frac{-1}{9}$$

$$\tilde{v}_\eta - \tilde{u}_\xi = \frac{2}{9}\xi + \frac{\eta}{3} \Rightarrow \tilde{v}_{\eta\xi} - \tilde{u}_{\xi\xi} = \frac{2}{9}$$

We also have

$$\tilde{u}_{\eta\xi} + \tilde{v}_{\xi\xi} = 0$$

$$\tilde{v}_{\eta\eta} - \tilde{u}_{\eta\xi} = \frac{1}{3} \qquad \Rightarrow \boxed{\tilde{v}_{\eta\eta} + \tilde{v}_{\xi\xi} = \frac{1}{3}}$$

(S)

Problem 5: Let Ω an open bounded domain of $\mathbb{R}^3$, with smooth boundary $\partial\Omega$.

1.) Consider the Dirichlet problems

$(5pts)$

$$-\Delta u = 1, \quad (x,y,z) \in \Omega, \tag{0.8}$$
$$u|_{\partial\Omega} = 0. \tag{0.9}$$

Give a function $r = r(x,y,z)$ such that $v = r + u$ satisfies the problem

$$-\Delta v = 0, \quad (x,y,z) \in \Omega, \tag{0.10}$$
$$v|_{\partial\Omega} = g, \tag{0.11}$$

where g is continuous.

2.) Based on the potential theory, we can construct a solution $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ of (0.10)-(0.10) of the form

$$v(x) = -\iint_{\partial\Omega} h(y)\frac{\partial G}{\partial n}(x-y)\,d\sigma_y \tag{0.12}$$

where σ_y is the surface area, $y \in \partial\Omega$, $G(x) = \frac{1}{4\pi|x|}$ is the fundamental solution of (0.10) but for $(x,y,z) \in \mathbb{R}^3$.

$(5pts)$ a.) Explain how the expression of G is obtained?

$(10pts)$ b.) Explain how h has to be chosen.

Solution:

1.) $-\Delta V = -\Delta u - \Delta r = 0 \implies \Delta r = 1$

$\underbrace{\quad}_{=1}$

So, we can take $r = \frac{x^2}{2}$ or $\frac{x^2+y^2}{4}$, or $\frac{x^2+y^2+z^2}{6}$

2.) a) We set $r = |x| = \sqrt{x^2+y^2+z^2}$; $x = (x,y,z)$

$V(x) = \tilde{V}(r) \implies \frac{\partial V}{\partial x} = \frac{\partial \tilde{V}}{\partial r}\frac{\partial r}{\partial x}$

$\frac{\partial^2 V}{\partial x^2} = \tilde{V}_{rr}\frac{x^2}{r^2} - \tilde{V}_r\frac{x^2}{r^3} + \frac{1}{r}\tilde{V}_r$

$-\Delta V = 0 \implies \frac{2}{r}\tilde{V}_r + \tilde{V}_{rr} = 0 \implies \tilde{V}(r) = \frac{b}{r} + C, \; r > 0$

We also have $-\Delta G = \delta_0$ in $\mathbb{R}^3$. $\implies C = 0, \; b = \frac{1}{4\pi}$

(Dirac measure)

b) $g(x_0) = \frac{1}{2}h(x_0) - \iint_{\partial\Omega} h(y)\frac{\partial G}{\partial n}(x_0-y)\,d\sigma_y$

(6)