MATH680 — Dynamic Programming
Final Exam
Department of Mathematics
KFUPM

Duration: 2h30

Instructions

Answer **all five problems**.
Show all reasoning;
numerical precision to 2–3 decimals is sufficient.

Problem 1 — 5G Network Slicing Admission & Pricing MDP (20 Points)

We consider a finite-horizon MDP with capacity $C = 2$, states $x \in \{0, 1, 2\}$, survival probability $s = 0.5$, and horizon $T = 2$ (stages $k = 0, 1$; $J_2 \equiv 0$).

Arrival scenarios:

$$A \in \{1, 2\}, \qquad P(A = 1) = 0.6,\ P(A = 2) = 0.4.$$

Decision at each k: choose price $p \in \{1, 3\}$ and admission $u \in \{0, 1\}$. Reward:

$$r(x, p, u; A) = p \cdot \min(A,\, C - x)\, u.$$

State transition (expectation approximation):

$$x' = \min\big(sx + u \min(A, C - x),\, C\big).$$

Discount factor: $\gamma = 0.9$.

1. Write explicitly the Bellman equations for $J_1(x)$ and $J_0(x)$.
2. Compute numerical values of $J_1(0), J_1(1), J_1(2)$.
3. Determine, for each $x \in \{0, 1, 2\}$, the optimal pair (p^*, u^*) at stage $k = 0$.
4. Explain qualitatively how replacing the high price $p = 2$ of the classroom example by $p = 3$ shifts the pricing threshold.

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Problem 2 — Multi-Product Inventory DP with Capacity Constraint (20 Points)

We revisit the two-product inventory DP over periods $k = 0, 1$ (terminal time $k = 2$) with initial state

$$X_0 = (x_0^1, x_0^2) = (1, 0).$$

Demands:

$$w_k^1 \in \{0, 1\}, \quad P(0) = 0.3,\ P(1) = 0.7,$$
$$w_k^2 \in \{0, 1, 2\}, \quad P(0) = 0.2,\ P(1) = 0.5,\ P(2) = 0.3.$$

Ordering costs: $c^1 = 1$, $c^2 = 2$. Holding costs: $h^1 = h^2 = 1$.

Capacity constraint:

$$x_k^1 + x_k^2 + u_k^1 + u_k^2 - w_k^1 - w_k^2 \le C = 3 \quad \text{for each period } k.$$

Equivalently, after demand realization,

$$x_{k+1}^i = \max(0,\ x_k^i + u_k^i - w_k^i), \qquad i = 1, 2.$$

Terminal cost:

$$J_2(x^1, x^2) = (x_2^1)^2 + (x_2^2)^2.$$

1. Write the Bellman equation for $J_1(x)$ with the corrected capacity constraint and full expectation over (w^1, w^2).
2. For state $x_1 = (1, 1)$, compute the optimal order vector (u_1^1, u_1^2).
3. Compute $J_0(1, 0)$ by enumerating all feasible (u_0^1, u_0^2) satisfying the capacity constraints.
4. Briefly explain how the constraint $x_k^1 + x_k^2 + u_k^1 + u_k^2 - w_k^1 - w_k^2 \le 3$ creates coupling between the two products and affects optimal ordering.

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Problem 3 — Dynamic Programming for FX Trading and Hedging (20 Points)

We consider a two-period speculative trading problem on a single foreign currency (EUR), priced in USD.

Initial position:

$$C_0 = 100, \qquad P_0 = 0, \qquad S_0 = 1.00.$$

Two equiprobable scenarios for the FX rate:

$$\text{Scenario A: } (S_1, S_2) = (0.95, 0.90), \qquad \text{Scenario B: } (S_1, S_2) = (1.05, 1.20).$$

At each time $t = 0, 1$ we choose a trade quantity x_t (in EUR units):

- $x_t > 0$: buy x_t EUR at price S_t,
- $x_t < 0$: sell $|x_t|$ EUR at price S_t.

Resource / feasibility constraints:

- No short-selling in EUR:

$$P_t + x_t \geq 0 \quad \Longrightarrow \quad x_t \geq -P_t.$$

- No borrowing in cash (USD):

$$C_t - x_t S_t \geq 0 \quad \Longrightarrow \quad x_t \leq \frac{C_t}{S_t}.$$

So the feasible set at time t is the interval

$$x_t \in \left[-P_t,\ \frac{C_t}{S_t}\right].$$

State dynamics:

$$P_{t+1} = P_t + x_t, \qquad C_{t+1} = C_t - x_t S_t.$$

At $t = 2$, all remaining inventory is liquidated:

$$W_2 = C_2 + P_2 S_2.$$

No transaction costs, no spread, and discount factor $\beta = 1$.

1. Write the Bellman equations for $Q_1(P_1, C_1)$ and for Q_0 (as functions of x_0).
2. At $t = 1$, and for a given feasible state (P_1, C_1) in each scenario separately, derive the optimal action (buy as much as possible, sell as much as possible, or hold) using monotonicity of W_2 in x_1.
3. Determine the optimal initial purchase x_0 at $t = 0$.
4. Explain how the asymmetry between scenarios A (down-down) and B (up-up) alters the structure relative to the classroom example.

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Problem 4 — TSP-MTW-RC with Dynamic Customer Insertion (20 Points)

We start with the depot (0) and three customers $\{1, 2, 3\}$. Travel times equal Euclidean distances:

$$d_{01} = 14.4, \quad d_{12} = 9.9, \quad d_{23} = 10.2, \quad d_{03} = 12.5, \quad d_{13} = 11.7, \quad d_{02} = 15.8.$$

Each customer i has a single time window $[L_i, U_i]$:

$$\begin{aligned}
&\text{Depot: } [540, 1080],\\
&1 : [580, 640],\\
&2 : [600, 660],\\
&3 : [620, 700].
\end{aligned}$$

Service times: $s_i = 5$. Resource consumption γ_{ij} is given as in tables in class (you may assume values consistent with the project, or symbolic γ_{ij} if preferred). Resource capacity: $R_{\max} = 12$. The vehicle departs the depot at time $t'_0 = 560$.

The initial static DP solution determines an optimal order to visit $\{1, 2, 3\}$. After the vehicle has started executing the route (i.e., after reaching customer 1), a new request arrives from customer 4 with:

$$d_{i4} \text{ and } d_{4j} \text{ known}, \qquad \text{time window } [L_4, U_4], \qquad \text{service time } s_4 = 5.$$

You must determine whether inserting customer 4 is feasible, and if so, the best insertion point.

1. Write the DP recurrence
$$f(S, j, t'_j, r_j)$$
for this problem (specialized to $|S| \leq 4$).

2. Compute feasibility (arrival times, waiting times, remaining resources) of the *initial* route visiting $\{1, 2, 3\}$.

3. Customer 4 arrives dynamically while the vehicle is at customer 1. Check all insertion positions:
$$1 \to 4 \to 2 \to 3, \qquad 1 \to 2 \to 4 \to 3, \qquad 1 \to 2 \to 3 \to 4,$$
and determine which are feasible.

4. Among feasible insertions, compute the incremental cost
$$\Delta C = (\text{travel} + \text{waiting} + \text{service})_{\text{new}} - (\text{travel} + \text{waiting} + \text{service})_{\text{old}},$$
and identify the best insertion point.

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Problem 5 — Smart Grid Pricing MDP (Single Class, Three States) (20 Points)

Single load class with capacity states $b \in \{0, 1, 2\}$.

Demand:

$$d_t \sim \text{Poisson}(\Lambda(u)), \qquad \Lambda(u) = 3e^{-u \ln 2}.$$

Maximum served:

$$X_t = \min(d_t,\, b_t).$$

Net price (linear energy cost):

$$\tilde{u} = u - 0.2, \qquad u \in \{1, 2\}.$$

Transition:

$$b_{t+1} = b_t - X_t,$$

(no returning capacity). Discount factor: $\gamma = 0.9$. Horizon: $T = 2$ with $V_2 \equiv 0$.

1. Compute $E[X \mid b = 1, u]$ and $E[X \mid b = 2, u]$ for $u = 1, 2$.
2. Write the Bellman equations for $V_1(b)$ and $V_0(b)$.
3. Compute the optimal policy at $t = 1$ for all b.
4. Compute $V_0(2)$ and determine the optimal u_0^*.
5. Explain why large-capacity states sometimes prefer a higher price and sometimes prefer a lower price, depending on elasticity.

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