MATH680: Dynamic Programming
Midterm Exam - Term 251
Department of Mathematics
KFUPM

Name:[a]

[a] *Id:*
[b] *12th October 2025*

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1. Task Scheduling (10 points)

You manage a single machine and must schedule a set of tasks $J = \{1, 2, 3, 4\}$. Each task j has a processing time p_j. There are *sequence-dependent setup costs* $s_{i\to j}$ incurred when task j is performed immediately after task i. Use a dummy start task 0 with $p_0 = 0$ and setup $s_{0\to j}$ to model the first-task setup. The goal is to *minimize*

$$\sum_{j\in J} C_j^2 \ + \sum_{(i\to j)\ \text{adjacent in the sequence}} s_{i\to j},$$

where C_j is the completion time of task j.

Processing times:

$$(p_1, p_2, p_3, p_4) = (4,\ 2,\ 3,\ 4).$$

Precedence constraints:

$$1 \to 4, \qquad 2 \to 3.$$

Sequence-dependent setup costs $s_{i\to j}$ for $i \in \{0, 1, \ldots, 4\}$, $j \in \{1, \ldots, 4\}$ (dash "–" means not applicable):

$s_{i\to j}$	1	2	3	4
0	4	3	3	2
1	–	2	4	3
2	3	–	5	4
3	2	5	–	6
4	4	4	7	–

1. **Formulate a subset Dynamic programming approach to solve this problem.**
2. **Recover the optimal schedule using the DP algorithm.**
3. **Provide:**
   (a) The optimal job sequence satisfying the precedences;
   (b) The minimal value of $\sum C_j^2 + \sum s_{i\to j}$;
   (c) A breakdown of the total setup cost and the sum of completion times.

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2. Optimal Control Problem (Discrete-Time, Finite Horizon) (10 points)

Consider the scalar, discrete-time linear system

$$x_{k+1} \;=\; a\,x_k \;+\; b\,u_k, \qquad a = 0.8,\;\; b = 1.0,$$

for stages $k = 0, 1, \ldots, N-1$ with horizon $N = 3$ and initial condition

$$x_0 = 1.$$

For the given weights

$$q = 1, \qquad r = 0.2, \qquad P_N = p = 3,$$

the performance index is

$$J(x_{0:N}, u_{0:N-1}) \;=\; x_N^2\,p \;+\; \sum_{k=0}^{N-1} \Big(q\,x_k^2 \;+\; r\,u_k^2\Big).$$

The control sequence is $u_0, u_1, \ldots, u_{N-1}$, and the resulting state sequence is $x_1, \ldots, x_N$.

1. Define the value function $J_k(x)$ (the minimal future cost from stage k with state x) and write the Bellman recursion

$$J_k(x) \;=\; \min_u \Big\{\, q\,x^2 + r\,u^2 + J_{k+1}(ax + bu) \,\Big\}, \qquad J_N(x) = p\,x^2.$$

2. Compute (numerically) the optimal feedback control $u_k^*(x)$ by backward induction and report $J_0(x_0)$. **Solve numerically: discretize the state and use $u_k^* = -K_k x_k$.**
3. Plot the closed-loop trajectory x_k under u_k^* for $k = 0, \ldots, N$ starting from $x_0 = 1$.

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3. Pontryagin Minimum Principle (10 points)

Consider the dynamic system:

$$\dot{x}(t) = -\alpha\, x(t) + u(t), \qquad \alpha > 0,$$

over a fixed horizon $t \in [0, T]$.

The initial state is fixed,

$$x(0) = x_0,$$

and the terminal state $x(T)$ is *free*.

We would like to minimize the quadratic effort and terminal deviation:

$$J \;=\; \frac{1}{2}\, q_f\, x(T)^2 \;+\; \frac{1}{2}\int_0^T \big(q\, x(t)^2 + r\, u(t)^2\big)\, dt.$$

Take $\alpha = 0.2$, $q = 0.5$, $r = 0.25$, $q_f = 2$, $T = 3$, , and $x_0 = 1$.

1. Formulate the Hamiltonian $H\big(x, \lambda, u\big)$.
2. Derive the costate dynamics $\lambda(t)$ and boundary condition.
3. Find the expression of the minimizing control $u^*(t) \in \arg\min_u$.

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4. Dynamic Portfolio Analysis (10 points)

An investor with initial wealth $W_0 > 0$ allocates fractions (w, w_f) across one risky asset and one riskless asset, with the budget

$$w + w_f = 1.$$

Let $R_f > 0$ be the (gross) return of the riskless asset. The risky asset has a two-state (up/down) gross return:

$$R = \begin{cases} R^U & \text{with probability } p, \\ R^D & \text{with probability } 1-p, \end{cases}$$

Assume the two assets' returns are independent.

The terminal wealth at $t = 1$ is

$$W_1 \; = \; W_0\big(wR + w_f R_f\big).$$

The investor has a logarithmic utility and chooses (w, w_f) to

$$\max_{w, w_f} \; \mathbb{E}\big[\ln W_1\big] \quad \text{s.t.} \quad w + w_f = 1, \;\; \text{and (optionally) } w, w_f \geq 0.$$

1. Write $\mathbb{E}[\ln W_1]$.
2. Derive the First-order conditions.
3. For the numerical values below:

$$p = 0.52, \quad R_f = 1.03, \quad R^U = 1.13, \; R^D = 0.93.$$

   (i) Compute the state probabilities and W_1 in each state as functions of w.

   (ii) Maximize $\mathbb{E}[\ln W_1]$ numerically over w with $w_f = 1 - w$. Report the optimal weights and the value of $\mathbb{E}[\ln W_1]$.

4. Discuss how the optimal weights change as you vary p, or R_f. Explain the effect of increasing R_f on w_f and the trade-off between the two assets when p moves up or down. In particular, for which values of p, the risky asset would always be overlooked?

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