MATH 514 Comprehensive Exam

Q.1 (10 points) Evaluate the integral $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2(x^2+4)}\,dx$ using residue theory.

Hint: Consider $f(z) = \frac{e^{iz}}{(z^2+1)^2(z^2+4)}$.

Sol: Consider $f(z) = \frac{e^{iz}}{(z^2+1)^2(z^2+4)}$, $z = i$, $z = 2i$ in the upper half plane.

$$\operatorname{Res}[f(z); i] = \frac{1}{1!} \lim_{z\to i} \frac{d}{dz}\left[\frac{e^{iz}}{(z+i)^2(z^2+4)}\right]$$

$$= \lim_{z\to i} \frac{ie^{iz}(z+i)^2(z^2+4) - e^{iz}\left(2(z+i)(z^2+4) + 2z(z+i)^2\right)}{(z+i)^4(z^2+4)^2}$$

$$= \frac{ie^{-1}(-4)(3) - e^{-1}\left(2(2i)(3) + 2i(-4)\right)}{(16)(9)}$$

$$= \frac{-12e^{-1}i - 12ie^{-1} + 8e^{-1}i}{(16)(9)} = \frac{-16e^{-1}i}{(16)(9)}$$

$$= \frac{-e^{-1}i}{9}$$

$$\operatorname{Res}[f(z); 3i] = \lim_{z\to 2i} \frac{e^{iz}}{(z^2+1)^2(z+2i)} = \frac{e^{-2}}{9(4i)}$$

$$= \frac{-e^{-2}i}{36}$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2(x^2+4)}\,dx = 2\pi i\left[\frac{-e^{-1}i}{9} - \frac{e^{-2}i}{36}\right]$$

$$= 2\pi\left[\frac{1}{9e} + \frac{1}{36e^2}\right]$$

— : — : —

Q.2 (10 points) Use Laplace transform to evaluate the integral

$$\int_0^\infty \frac{\sin(xt)}{x(x^2+4)}dx$$

Sol:

Let $f(t) = \int_0^\infty \frac{\sin(xt)}{x(x^2+4)} dx$

Taking Laplace transform, we get

$$F(s) = \int_0^\infty \frac{1}{x(x^2+4)} \mathcal{L}\{\sin(xt)\} dx$$

$$= \int_0^\infty \frac{1}{x(x^2+4)} \frac{x}{s^2+x^2} dx = \int_0^\infty \frac{1}{(x^2+4)(x^2+s^2)} dx$$

$$= \frac{1}{s^2-4} \int_0^\infty \left[\frac{1}{x^2+4} - \frac{1}{x^2+s^2}\right] dx$$

$$= \frac{1}{s^2-4} \left[\frac{1}{2} \tan^{-1}\frac{x}{2} - \frac{1}{s} \tan^{-1}\frac{x}{s} \right]\Bigg|_0^\infty$$

$$= \frac{1}{s^2-4}\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{s}\frac{\pi}{2}\right] = \frac{\pi}{4(s^2-4)}\left[1 - \frac{2}{s}\right]$$

$$= \frac{\pi}{4(s^2-4)} \cdot \frac{s-2}{s} = \frac{\pi}{4s(s+2)} = \frac{\pi}{8}\left[\frac{1}{s} - \frac{1}{s+2}\right]$$

$$f(t) = \frac{\pi}{8}\left[1 - e^{-2t}\right]$$

—— : —— : ——

Q.3 (15 points) Use Laplace transform to solve the heat equation

$$ku_{xx} = u_t, \ 0 < x < L, \ t > 0$$

under the following conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$$

and

$$u(x,0) = x. \ \ 0 < x < L$$

Apply residue theory and write final answer as a series.

Sol: Taking Laplace transform, we get

$$K \frac{d^2U}{dx^2} = sU(x,s) - u(x,0)$$

$$\frac{d^2U}{dx^2} - \frac{s}{K} U = -\frac{x}{K}$$

$$u(0,t) = 0 \Rightarrow U(0,s) = 0 \text{ and } u(L,t) = 0 \Rightarrow U(L,s) = 0$$

To find U_c: $\frac{d^2U}{dx^2} - \frac{s}{K} U = 0, \qquad m = \pm \sqrt{\frac{s}{K}}$

$$U_c(x,s) = A \cosh \sqrt{\frac{s}{K}}\, x + B \sin \sqrt{\frac{s}{K}}\, x$$

To find U_p: Let $U_p = A + Bx$, then

$$-\frac{s}{K}(A + Bx) = -\frac{1}{K} x$$

$$\Rightarrow A = 0, \quad -\frac{s}{K} B = -\frac{1}{K} \quad \Rightarrow \quad B = \frac{1}{s}$$

$$U(x,s) = A \cosh \sqrt{\frac{s}{K}}\, x + B \sin \sqrt{\frac{s}{K}}\, x + \frac{x}{s}$$

$$U(0,s) = 0 \Rightarrow A = 0, \quad U(L,s) = 0 \Rightarrow B \sinh \sqrt{\frac{s}{K}}\, L + \frac{L}{s} = 0$$

$$B = -\frac{\frac{L}{s}}{\sinh \sqrt{\frac{s}{K}}\, L}$$

$$\therefore U(x,s) = -\frac{L}{\ } \cdot \frac{\sinh \sqrt{\frac{s}{K}}\, x}{\ } + \frac{x}{\ }$$

$$s \quad \sinh\sqrt{\frac{s}{K}}\,L$$

Check if $s=0$ is a singularity or not ?

$$U(x,s) = \frac{1}{s}\left\{ x - \frac{L\left[\sqrt{\frac{s}{K}}\,x + \frac{1}{3!}\left(\sqrt{\frac{s}{K}}\,x\right)^3 + \cdots\right]}{\sqrt{\frac{s}{K}}\,L + \frac{1}{3!}\left(\sqrt{\frac{s}{K}}\,L\right)^3 + \cdots} \right\}$$

$$= \frac{1}{s}\left[x - \frac{x + \frac{1}{3!}\,\frac{s}{K}\,x^3 + \cdots}{1 + \frac{1}{3!}\,\frac{s}{K}\,L^2 + \cdots} \right]$$

$$= \frac{1}{s}\left\{ x - \left(x + \frac{s x^3}{6K} + \cdots\right)\left(1 + \frac{sL^2}{6K} + \cdots\right)^{-1} \right\}$$

$$= \frac{1}{s}\left[x - \left(x + \frac{s x^3}{6K} + \cdots\right)\left(1 - \frac{sL^2}{6K} - \cdots\right) \right]$$

$$= \frac{1}{s}\left[x - \left(x - \frac{sL^2 x}{6K} - \cdots + \frac{s x^3}{6K} - \cdots\right) \right]$$

$$= \frac{1}{s}\left[\frac{sL^2 x}{6K} - \frac{s x^3}{6K} + O(s) \right]$$

$$= \frac{xL^2 - x^3}{6K} + O(s)$$

$s=0$ is a removable singularity.

Now $\sinh\sqrt{\frac{s}{K}}\,L = 0 \Rightarrow \sqrt{\frac{s}{K}}\,L = n\pi i$, $n = 1, 2, \cdots$

$$s = \frac{-Kn^2\pi^2}{L^2}, \quad n = 1, 2, 3, \cdots \quad \text{simple poles}$$

The residue of $U(x,s)\,e^{st}$ at these poles

$$\lim_{s \to -\frac{Kn^2\pi^2}{L^2}} \left(s + \frac{Kn^2\pi^2}{L^2}\right)\frac{e^{st}}{s}\left(x - \frac{L\sinh\sqrt{\frac{s}{K}}\,x}{\sinh\sqrt{\frac{s}{K}}\,L} \right)$$

$$= \lim_{s \to -\frac{Kn^2\pi^2}{L^2}} \frac{e^{st}}{s} \left[\lim_{s \to -\frac{Kn^2\pi^2}{L^2}} x \left(s + \frac{Kn^2\pi^2}{L^2}\right) - \lim_{s \to -\frac{Kn^2\pi^2}{L^2}} \frac{\left(s + \frac{Kn^2\pi^2}{L^2}\right) L \sinh \sqrt{\frac{s}{K}}\, x}{\sinh \sqrt{\frac{s}{K}}\, L} \right]$$

$$= \frac{e^{-\frac{Kn^2\pi^2}{L^2}t}}{-\frac{Kn^2\pi^2}{L^2}} \left[0 - L \sinh \frac{n\pi}{L} x i \cdot \lim_{s \to -\frac{Kn^2\pi^2}{L^2}} \frac{s + \frac{Kn^2\pi^2}{L^2}}{\sinh \sqrt{\frac{s}{K}}\, L} \right]$$

$$= -\frac{L^3 e^{-\frac{Kn^2\pi^2}{L^2}t}}{Kn^2\pi^2} i \sin \frac{n\pi}{L} x \cdot \lim_{s \to -\frac{Kn^2\pi^2}{L^2}} \frac{1}{\frac{L}{2\sqrt{sK}} \cosh \sqrt{\frac{s}{K}}\, L}$$

$$= \frac{-iL^3 e^{-\frac{Kn^2\pi^2}{L^2}t}}{Kn^2\pi^2} \sin \frac{n\pi}{L} x \cdot \frac{1}{\frac{L}{2\, \frac{Kn\pi}{L} i} \cosh \frac{n\pi}{L} L i}$$

$$= \frac{-iL^3 e^{-\frac{Kn^2\pi^2}{L^2}t}}{Kn^2\pi^2} \cdot \sin \frac{n\pi}{L} x \cdot \frac{\frac{2Kn\pi i}{L}}{L} \cdot \frac{1}{\cos n\pi}$$

$$= \frac{2L e^{-\frac{Kn^2\pi^2}{L^2}t}}{n\pi} \sin \frac{n\pi}{L} x \, (-1)^n$$

$$\therefore u(x,t) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-\frac{Kn^2\pi^2}{L^2}t} \sin \frac{n\pi}{L} x$$

——— : ——— : ———

Q.4 (10 points) Solve the integral equation using Fourier transform

$$\int_{-\infty}^{\infty} e^{-4t^2} f(x-t)dt = e^{-3x^2}.$$

Sol: $e^{-4x^2} * f(x) = e^{-3x^2}$

$\Rightarrow$ Applying Fourier transform

$$\sqrt{\frac{\pi}{4}}\, e^{-\frac{\alpha^2}{16}} F(\alpha) = \sqrt{\frac{\pi}{3}}\, e^{-\frac{\alpha^2}{12}}$$

$$\Rightarrow F(\alpha) = \frac{2}{\sqrt{3}}\, e^{\frac{\alpha^2}{16}-\frac{\alpha^2}{12}} = \frac{2}{\sqrt{3}}\, e^{-\frac{\alpha^2}{4}\left(\frac{1}{12}\right)}$$

$$f(x) = \frac{1}{2\pi}\,\frac{2}{\sqrt{3}} \int_{-\infty}^{\infty} e^{-\frac{\alpha^2}{4}\left(\frac{1}{12}\right)}\, e^{i\alpha x}\, d\alpha$$

$$= \frac{1}{\pi\sqrt{3}} \int_{-\infty}^{\infty} e^{-\frac{1}{48}\left(\alpha^2 - 48\alpha x i\right)}\, d\alpha$$

$$= \frac{1}{\pi\sqrt{3}} \int_{-\infty}^{\infty} e^{-\frac{1}{48}\left[\alpha^2 - 2\alpha(24xi) + (24xi)^2 - (24xi)^2\right]}\, d\alpha$$

$$= \frac{1}{\pi\sqrt{3}} \int_{-\infty}^{\infty} e^{-\frac{1}{48}\left[\alpha - 24xi\right]^2 + \frac{1}{48}(24xi)^2}\, d\alpha$$

$$= \frac{1}{\pi\sqrt{3}}\, e^{-12x^2} \int_{-\infty}^{\infty} e^{-\frac{1}{48}u^2}\, du,$$

$$u = \alpha - 24xi$$

$$= \frac{1}{\pi\sqrt{3}}\, e^{-12x^2} \cdot \frac{\sqrt{\pi}}{\sqrt{\frac{1}{48}}} = \frac{1}{\sqrt{3}}\, 4\sqrt{3}\, e^{-12x^2} = 4e^{-12x^2}$$

Q.5 (10 points) Use Fourier transform to solve the heat equation

$$ku_{xx} = u_t, \ -\infty < x < \infty, \ t > 0$$

under the following conditions $u(x,0) = f(x)$ where $f(x) = \begin{cases} u_o & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$.

Sol: Taking Fourier transform, we get

$$-K\alpha^2 U(\alpha,t) = \frac{dU}{dt} \Rightarrow U(\alpha,t) = A e^{-K\alpha^2 t}$$

$$u(x,0) = f(x) \Rightarrow U(\alpha,0) = \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$$

$$U(\alpha,0) = \int_{-1}^{1} u_o e^{-i\alpha x} dx = \frac{u_o}{-i\alpha} e^{-i\alpha x} \Big|_{-1}^{1}$$

$$= -\frac{u_o}{i\alpha}\left(e^{-i\alpha} - e^{i\alpha}\right) = \frac{2u_o}{\alpha}\left[\frac{e^{i\alpha} - e^{-i\alpha}}{2i}\right]$$

$$= \frac{2u_o}{\alpha} \sin\alpha$$

$$\Rightarrow A = \frac{2u_o}{\alpha} \sin\alpha$$

$$U(\alpha,t) = \frac{2u_o}{\alpha} \sin\alpha \, e^{-K\alpha^2 t}$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2u_o \frac{\sin\alpha}{\alpha} e^{-K\alpha^2 t} e^{i\alpha x} d\alpha$$

$$= \frac{u_o}{\pi}\int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha} e^{-K\alpha^2 t} [\cos\alpha x + i\sin\alpha x] d\alpha$$

$$= \frac{u_o}{\pi}\int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha} \cos\alpha \, e^{-K\alpha^2 t} d\alpha + 0$$

Q.6 (10 points) Solve using Mellin transform

$$x^2 u_{xx} + x u_x + y_{yy} = 0, \quad 0 \le x < \infty, \quad 0 < y < 1$$

under the conditions $u(x,0) = 0$ and $u(x,1) = \begin{cases} A & 0 < x < 1 \\ 0 & x > 1 \end{cases}$.

Sol: Taking Mellin transform, we get

$$p(1+p)\tilde{u}(p,y) - p\,\tilde{u}(p,y) + \frac{d^2 u}{dy^2} = 0$$

$$\Rightarrow \frac{d^2\tilde{u}}{dy^2} + p^2\tilde{u} = 0$$

$$\tilde{u}(p,y) = C_1 \cos py + C_2 \sin py$$

$$u(x,0) = 0 \Rightarrow \tilde{u}(p,0) = 0 \Rightarrow C_1 = 0$$

$$u(x,1) = A\,H(1-x) \Rightarrow \tilde{u}(p,1) = \int_0^\infty x^{p-1} A\,H(1-x)\,dx$$

$$\tilde{u}(p,1) = A\int_0^1 x^{p-1}\,dx = \frac{A}{p} x^p \Big|_0^1 = \frac{A}{p}$$

$$\Rightarrow C_2 \sin p = \frac{A}{p} \Rightarrow C_2 = \frac{A}{p \sin p}$$

$$\tilde{u}(p,y) = \frac{A}{p\sin p} \sin py$$

$$u(x,y) = \frac{A}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-p}\,\frac{\sin py}{p\sin p}\,dp$$

—— : —— : ——

Q.7 (10 points) Show the Hankel transform

$$\mathcal{H}_o\{e^{-ar}\} = \frac{a}{(a^2+\alpha^2)^{3/2}}$$

Sol: $\mathcal{H}_0\{e^{-ar}\} = \int_0^\infty r\, J_0(\alpha r)\, e^{-ar}\, dr$

$$= \int_0^\infty \frac{t}{\alpha} J_0(t)\, e^{-\frac{a}{\alpha}t} \frac{1}{\alpha} dt, \qquad \alpha r = t$$

$$= \frac{1}{\alpha^2} \int_0^\infty t\, J_0(t)\, e^{-st} dt \quad , \qquad s = \frac{a}{\alpha}$$

$$= \frac{1}{\alpha^2} \mathcal{L}\{t\, J_0(t)\}$$

Since $\mathcal{L}\{t^v J_v(t)\} = \dfrac{2^v \Gamma(v+\frac{1}{2})}{\sqrt{\pi}\, (1+s^2)^{v+\frac{1}{2}}}$

for $v=0$, $\mathcal{L}\{J_0(t)\} = \dfrac{\Gamma(\frac{1}{2})}{\sqrt{\pi}\,(1+s^2)^{\frac{1}{2}}} \qquad \dfrac{1}{(1+s^2)^{\frac{1}{2}}}$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$\mathcal{H}_0\{e^{-ar}\} = \frac{1}{\alpha^2}(-1)\frac{d}{ds}\{\mathcal{L}\{J_0(t)\}\}$$

$$= \frac{-1}{\alpha^2} \frac{d}{ds}\left(\frac{1}{(1+s^2)^{\frac{1}{2}}}\right) = -\frac{1}{\alpha^2}\left(-\frac{1}{2}\right)(1+s^2)^{-\frac{3}{2}} \cdot 2s$$

$$= \frac{1}{\alpha^2} \frac{s}{(1+s^2)^{3/2}} = \frac{1}{\alpha^2} \frac{\frac{a}{\alpha}}{\left(1+\frac{a^2}{\alpha^2}\right)^{3/2}} = \frac{a}{(a^2+\alpha^2)^{3/2}}$$

—— : —— : ——

Q.8 (10 points) Solve the axisymmetric bihormonic equation using Hankel transform

$$\begin{aligned}\nabla^4 u(r,z) &= 0, \quad 0 \le r < \infty, \quad z > 0\\ u(r,0) &= f(r), \quad 0 \le r < \infty,\\ \frac{\partial u}{\partial z} &= 0 \ \ on \ \ z = 0, \quad 0 \le r < \infty\\ u(r,z) &\to 0 \ \ as \ \ z \to \infty.\end{aligned}$$

Sol: $\nabla^2 u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{\partial^2 u}{\partial z^2}$

$$\nabla^4 u = \nabla^2 \nabla^2 u = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) u$$

$$= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 u$$

$$\mathcal{H}_0\{u(r,z)\} = \tilde{u}_0(\alpha, z)$$

and $\mathcal{H}_0\left\{\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}\right\} = -\alpha^2 \tilde{u}_0(\alpha, z)$

$$\Rightarrow \left(\frac{d^2}{dz^2} - \alpha^2\right)\tilde{u}_0(\alpha, z) = 0 \quad , \quad z > 0$$

Roots of the auxiliary equation are $\pm\alpha, \pm\alpha$

$$\tilde{u}_0(\alpha, z) = (A + Bz)e^{-\alpha z} + (C + Dz)e^{\alpha z}$$

The Solution is bounded $\Rightarrow$ Put $C = D = 0$

So $\tilde{u}_0(\alpha, z) = (A + Bz)e^{-\alpha z}$

Now $u(r,0) = f(r) \Rightarrow \tilde{u}_0(\alpha, 0) = \tilde{f}(\alpha)$

$$\frac{\partial u(r,0)}{\partial z} = 0 \quad \Rightarrow \frac{d}{dz}\tilde{u}(\alpha, 0) = 0$$

$$\frac{d\tilde{u}_0}{dz} = -\alpha(A + Bz)e^{-\alpha z} + Be^{-\alpha z}$$

$\tilde{u}_0(\alpha, 0) = \tilde{f}(\alpha) \Rightarrow A = \tilde{f}(\alpha)$

$$\frac{d\tilde{u}_0(\alpha,0)}{dz} = 0 \quad \Rightarrow \quad -\alpha A + B = 0 \Rightarrow B = \alpha \tilde{f}(\alpha)$$

$$\tilde{u}_0(\alpha, z) = \tilde{f}(\alpha)\,(1+\alpha z)\,e^{-\alpha z}$$

Taking inverse Hankel transform, we get

$$u(r,z) = \int_0^\infty \alpha J_0(\alpha r)\,\tilde{f}(\alpha)\,(1+\alpha z)\,e^{-\alpha z}\,d\alpha$$

— : — : —

Q.9 (15 points) Solve the integral equation using Wiener-Hopf technique

$$\int_0^\infty e^{-|x-\xi|}u(\xi)d\xi = -\frac{1}{4}u(x)+1, \quad 0<x<\infty.$$

$u(x)$ is bounded as $x \to \infty$.

Sol: Note that $u(x) = 4 - 4\int_0^\infty e^{-|x-\xi|}u(\xi)d\xi$ is bounded as $x \to \infty$. Thus $u_+^*(\alpha)$ is analytic function of α for $Im(\alpha)>0$.

Here $K(x-\xi) = e^{-|x-\xi|}$, $\mu = -\frac{1}{4}$, $f(x) = 1$, $0<x<\infty$

Define $f_+(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$

and $f_+^*(\alpha) = \int_{-\infty}^\infty f_+(x)\,e^{-i\alpha x}\,dx = \int_0^\infty e^{-i\alpha x}\,dx$

$$= -\frac{1}{i\alpha}\,e^{-i\alpha x}\Big|_0^\infty = \frac{1}{i\alpha} = -\frac{i}{\alpha},$$

$$K(x) = e^{-|x|} \quad \text{and} \quad \mathcal{F}\{K(x)\} = \frac{2}{1+\alpha^2} = K^*(\alpha)$$

To extend the problem to $(-\infty, \infty)$, we write

$$\int_{-\infty}^{\infty} e^{-|x-\xi|} u(\xi)\, d\xi = \begin{cases} -\frac{1}{4} u(x) + f(x) & 0 < x < \infty \\ g(x) & -\infty < x < 0 \end{cases}$$

where $u(x) = 0$ and $f(x) = 0$ for $x < 0$

$g(x) = 0$ for $x > 0$

So $u(x) = u_+(x)$, $f(x) = f_+(x)$, $g(x) = g_-(x)$

and $\int_{-\infty}^{\infty} e^{-|x-\xi|} u_+(\xi)\, d\xi = -\frac{1}{4} u_+(x) + 1 + g_-(x)$

$$\overset{\mathcal{F}}{\Rightarrow} \quad \frac{2}{1+\alpha^2} u_+^*(\alpha) = -\frac{1}{4} u_+^*(\alpha) - \frac{i}{\alpha} + g_-^*(\alpha)$$

$$\frac{9+\alpha^2}{4(1+\alpha^2)} u_+^*(\alpha) = -\frac{i}{\alpha} \quad g_-^*(\alpha)$$

$$\frac{(\alpha+3i)(\alpha-3i)}{4(\alpha+i)(\alpha-i)} u_+^*(\alpha) = -\frac{i}{\alpha} + g_-^*(\alpha)$$

$$\frac{\alpha+3i}{4(\alpha+i)} u_+^*(\alpha) = \frac{-i(\alpha-i)}{\alpha(\alpha-3i)} + \frac{\alpha-i}{\alpha-3i} g_-^*(\alpha)$$

$$\frac{-i(\alpha-i)}{\alpha(\alpha-3i)} = \frac{-i}{3\alpha} - \frac{2i}{3(\alpha-3i)} \quad \text{using partial fractions}$$

$P_+(\alpha) = -\frac{i}{3\alpha}$ is analytic for $Im(\alpha) < 0$

$P_-(\alpha) = \frac{-2i}{3(\alpha-3i)}$ is analytic for $Im(\alpha) < 3$

$$\frac{\alpha+3i}{4(\alpha+i)} u_+^*(\alpha) + \frac{i}{3\alpha} = \frac{-2i}{3(\alpha-3i)} + \frac{\alpha-i}{\alpha-3i} g_-^*(\alpha)$$

LHS is analytic in the upper half plane $Im(\alpha) > 0$

RHS is analytic in the lowe half plane $Im(\alpha) < 3$

By the Liouville's theorem, both sides define an entire function $E^*(\alpha)$ and LHS $\to 0$ as $\alpha \to \alpha$

and RHS $\to 0$ as $\alpha \to -\infty$.

Thus $E^*(\alpha) \equiv 0$

$$\frac{\alpha+3i}{4(\alpha+i)} u_+^*(\alpha) + \frac{i}{3\alpha} = 0$$

$$u_+^*(\alpha) = \frac{-4i(\alpha+i)}{3\alpha(\alpha+3i)}$$

$$u(x) = \frac{1}{2\pi}\int_{ai-\infty}^{ai+\infty} u_+^*(\alpha) e^{i\alpha x} d\alpha$$

$$= -\frac{2}{3\pi} i \int_{ai-\infty}^{ai+\infty} \frac{\alpha+i}{\alpha(\alpha+3i)} e^{i\alpha x} d\alpha$$

Poles of $h(\alpha) = \dfrac{\alpha+i}{\alpha(\alpha+3i)}$ are at $\alpha = 0$, $\alpha = -3i$

$$Res[h(\alpha), \alpha=0] = \lim_{\alpha\to 0} \frac{\alpha+i}{\alpha+3i} e^{i\alpha x} = \frac{1}{3}$$

$$Res[h(\alpha), \alpha=-3i] = \lim_{\alpha\to -3i} \frac{\alpha+i}{\alpha} e^{i\alpha x} = \frac{2}{3} e^{3x}$$

$$u(x) = -\frac{2i}{3\pi}\cdot 2\pi i \left[\frac{1}{3} + \frac{2}{3} e^{3x}\right] = \frac{4}{9}\left(1 + 2e^{3x}\right)$$

—— : —— : ——