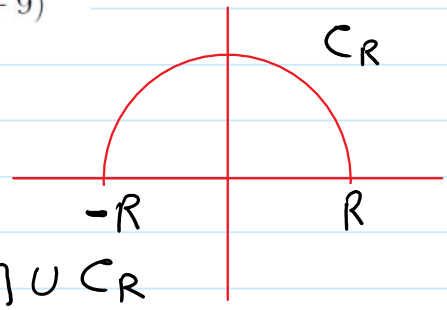
King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 514 Comprehensive Exam

Time Allowed: 180 Minutes

Q.1 (10 points) Evaluate the integral $\int_{-\infty}^{\infty} \frac{x\sin(\pi x)}{(x^2+4)(x^2+9)}dx$



Sol: Consider the integral

$$\int_{\Gamma} \frac{z\, e^{i\pi z}}{(z^2+4)(z^2+9)}\, dz, \quad \Gamma = [-R, R] \cup C_R$$

Poles of the integrand are $z = \pm 2i$, $\pm 3i$

Only $z = 2i$ and $z = 3i$ lie in the upper half plane.

$$\text{Res}\left[f(z);\ z = 2i\right] = \lim_{z \to 2i} \frac{z e^{i\pi z}}{(z+2i)(z^2+9)} = \frac{2i\, e^{-2\pi}}{4i(5)} = \frac{e^{-2\pi}}{10}$$

$$\text{Res}[f(z);\ z = 3i] = \lim_{z \to 3i} \frac{z e^{i\pi z}}{(z^2+4)(z+3i)} = \frac{3i\, e^{-3\pi}}{-5(6i)} = \frac{-e^{-3\pi}}{10}$$

$$\int_{\Gamma} \frac{z e^{i\pi z}}{(z^2+4)(z^2+9)}\, dz = \int_{C_R} \frac{z e^{i\pi z}}{(z^2+4)(z^2+9)}\, dz + \int_{-R}^{R} \frac{x e^{i\pi x}}{(x^2+4)(x^2+9)}\, dx$$

As $R \to \infty$, $\int_{C_R} \frac{z e^{i\pi z}}{(z^2+4)(z^2+9)}\, dz \to 0$

$$\int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{(x^2+4)(x^2+9)}\, dx = \frac{\pi i}{5}\left[e^{-2\pi} - e^{-3\pi}\right] \Rightarrow \int_{-\infty}^{\infty} \frac{x \sin(\pi x)\, dx}{(x^2+4)(x^2+9)} = \frac{\pi}{5}\left(e^{-2\pi} - e^{-3\pi}\right)$$

Q.2 (10 points) Use Laplace transform to solve the integral equation

$$f(t) = t\sin(t) + 2\int_0^t f'(\tau)\sin(t-\tau)d\tau, \quad f(0) = 0$$

Sol:

$$f(t) = t\sin t + 2 f'(t) * \sin t$$

$$F(s) = -\frac{d}{ds}\frac{1}{s^2+1} + [2sF(s) - f(0)]\frac{1}{s^2+1}$$

$$\left[1 - \frac{2s}{s^2+1}\right]F(s) = \frac{2s}{(s^2+1)^2} \Rightarrow F(s) = \frac{2s}{(s-1)^2(s^2+1)}$$

$$\frac{2s}{(s-1)^2(s^2+1)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+1}$$

$$2s = A(s-1)(s^2+1) + B(s^2+1) + Cs(s-1)^2 + D(s-1)^2$$

Put $s = 1$, $\quad 2 = 2B \quad\quad B = 1$

Compare the coefficients

$$2s = A(s^3 - s^2 + s - 1) + B(s^2+1) + C(s^3 - 2s^2 + s)$$
$$D(s^2 - 2s + 1)$$

$s^3 \quad 0 = A + C \quad\quad 1 - 2C + C = 0$

$C = 0$

$s^2 \quad 0 = -A + B - 2C + D \quad\quad A = B - 2C = 1 - 2C$

$s \quad 2 = A + C - 2D \quad\quad A = 0$

$2D = \underline{A + C} - 2 = -2 \quad\quad D = -1$

$$F(s) = \frac{1}{(s-1)^2} - \frac{1}{s^2+1}, \quad f(t) = te^t - \sin t$$

Q.3 (10 points) Use Laplace transform to solve the wave equation

$$a^2 u_{xx} = u_{tt} + b\sin(2t), \ \ x > 0, \ \ t > 0$$

under the following conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0, \quad x > 0$$

and

$$u(x,0) = 0 \qquad \lim_{x\to\infty} |u(x,t)| < \infty$$

Sol: $a^2 \frac{d^2U}{dx^2} = s^2 U(x,s) - s\,u(\cancel{x},0) - u_t(\cancel{x},0) + \frac{2b}{s^2+4}$

$\frac{d^2U}{dx^2} - \frac{s^2}{a^2} = \frac{b}{a^2} \cdot \frac{2}{s^2+4}$, $U_c = Ae^{\frac{s}{a}x} + Be^{-\frac{s}{a}x}$

Let $U_p = K$, then $0 - \frac{s^2}{a^2}K = \frac{b}{a^2}\frac{2}{s^2+4} \Rightarrow K = \frac{-2b}{s^2(s^2+4)}$

$$U(x,s) = Ae^{\frac{s}{a}x} + Be^{-\frac{s}{a}x} - \frac{2b}{s^2(s^2+4)}$$

$\lim\limits_{x\to\infty} |u(x,t)| < \infty \Rightarrow \lim\limits_{x\to\infty} |U(x,s)| < \infty \Rightarrow A = 0$

$u(x,0) = 0 \Rightarrow U(x,0) = 0 \Rightarrow B = \frac{2b}{s^2(s^2+4)}$

$$U(x,s) = \frac{2b}{s^2(s^2+4)}\left[e^{-\frac{s}{a}x} - 1\right]$$

using partial fractions

$$U(x,s) = \frac{b}{2}\left[\frac{1}{s^2} - \frac{1}{s^2+4}\right]\left[e^{-\frac{x}{a}s} - 1\right]$$

$$u(x,t) = \frac{b}{2}\left[\left(t-\frac{x}{a}\right) - \frac{1}{2}\sin 2\left(t-\frac{x}{a}\right)\right]H\left(t-\frac{x}{a}\right) - \frac{b}{2}t + \frac{b}{2}\sin 2t$$

Q.4 (10 points) Solve the integral equation for $f(x)$ using the Fourier transform

$$\int_{-\infty}^{\infty} f(t)f(x-t)dt = \frac{1}{x^2+a}.$$

Sol: $f(x) * f(x) = \dfrac{1}{x^2+a}$

$$\Rightarrow \quad F(\alpha)\,F(\alpha) = \mathcal{F}\left\{\frac{1}{x^2+a}\right\} = \sqrt{\frac{\pi}{2}}\ \frac{e^{-\sqrt{a}\,|\alpha|}}{\sqrt{a}}$$

$$(F(\alpha))^2 = \sqrt{\frac{\pi}{2a}}\ e^{-\sqrt{a}\,|\alpha|}$$

$$F(\alpha) = \left[\frac{\pi}{2a}\right]^{\frac{1}{4}} e^{-\frac{\sqrt{a}}{2}|\alpha|}$$

$$f(x) = \left(\frac{\pi}{2a}\right)^{\frac{1}{4}} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{\sqrt{a}}{2}|\alpha|}\, e^{i\alpha x}\, d\alpha$$

$$K \leftarrow \left(\frac{\pi}{2a}\right)^{\frac{1}{4}} \frac{1}{2\pi}$$

$$= \left(\frac{\pi}{2a}\right)^{\frac{1}{4}} \frac{1}{2\pi} \left[\int_{-\infty}^{0} e^{\left(\frac{\sqrt{a}}{2}+ix\right)\alpha} d\alpha + \int_{0}^{\infty} e^{-\left(\frac{\sqrt{a}}{2}-ix\right)\alpha} d\alpha\right]$$

$$= K\left[\left.\frac{e^{\left(\frac{\sqrt{a}}{2}+ix\right)\alpha}}{\frac{\sqrt{a}}{2}+ix}\right|_{-\infty}^{0} + \left.\frac{e^{-\left(\frac{\sqrt{a}}{2}-ix\right)\alpha}}{-\frac{\sqrt{a}}{2}+ix}\right|_{0}^{\infty}\right]$$

$$= K\left[\frac{1}{\frac{\sqrt{a}}{2}+ix} + \frac{1}{\frac{\sqrt{a}}{2}-ix}\right]$$

$$= K\ \frac{\sqrt{a}}{\frac{a}{4}+x^2} = K\ \frac{4\sqrt{a}}{a+4x^2}$$

Q.5 (13 points) Use appropriate Fourier transform to solve the Laplace equation

$$u_{xx} + u_{yy} = 0, \ x > 0, \ y > 0$$

under the following conditions $u(0, y) = k, \ \ y > 0$ and $u(x, 0) = 0, \ \ x > 0$. Solution is bounded as $x \to \infty$.

Sol: Apply Fourier Sine transform w.r.t x

$$-\alpha^2 U(\alpha, y) + \alpha u(0, y) + \frac{d^2 U}{dy^2} = 0$$

$$\frac{d^2 U}{dy^2} - \alpha^2 U = -K\alpha \ , \quad U_c(\alpha, y) = A e^{\alpha y} + B e^{-\alpha y}$$

Let $U_p = K$, then $-\alpha^2 C = -K\alpha \Rightarrow C = \frac{K}{\alpha}$

$$U(\alpha, y) = A e^{\alpha y} + B e^{-\alpha y} + \frac{K}{\alpha}$$

To keep the solution bounded, $A = 0$

$$U(\alpha, y) = B e^{-\alpha y} + \frac{K}{\alpha}$$

$$u(x, 0) = 0 \Rightarrow U(\alpha, 0) = 0 \Rightarrow B = -\frac{K}{\alpha}$$

$$U(\alpha, y) = -\frac{K}{\alpha} e^{-\alpha y} + \frac{K}{\alpha}$$

$$u(x, y) = \frac{2}{\pi} \int_0^\infty \frac{K}{\alpha} \left[1 - e^{-\alpha y}\right] \sin \alpha x \, d\alpha$$

$$= \frac{2K}{\pi} \int_0^\infty \frac{1}{\alpha} \sin \alpha x \, d\alpha - \frac{2K}{\pi} \int_0^\infty \frac{1}{\alpha} e^{-\alpha y} \sin \alpha x \, d\alpha$$

$$= \frac{2K}{\pi} \frac{\pi}{2} - \frac{2K}{\pi} \left(\frac{\pi}{2} - \tan^{-1} \frac{y}{x}\right) = \frac{2K}{\pi} \tan^{-1} \frac{x}{y}$$

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Q.6 (6+6 points) Use Mellin transform to show the following:

(a) $\mathcal{M}\{x^m e^{-nx}\} = \dfrac{\Gamma(m+p)}{n^{m+p-1}}$

(b) $\mathcal{M}\left\{\dfrac{1}{x^2+1}\right\} = \dfrac{\pi}{2}\csc\left(\dfrac{p\pi}{2}\right)$

Sol: (a) $M\{x^m e^{-nx}\} = \int_0^\infty x^{p-1}\, x^m e^{-nx}\, dx$

$$= \int_0^\infty x^{p+m-1} e^{-nx}\, dx$$

$$= \int_0^\infty \frac{t^{p+m-1}}{n^{p+m-1}} e^{-t} \frac{dt}{n} \qquad nx = t$$

$$= \int_0^\infty \frac{1}{n^{p+m}} t^{p+m-1} e^{-t}\, dt$$

$$= \frac{1}{n^{p+m}} \Gamma(p+m)$$

(b) Let $g(x) = \tan^{-1} x$, then $g'(x) = \dfrac{1}{x^2+1}$

$$M\{f(x)\} = M\{g'(x)\} = -(p-1)\,\tilde{g}(p-1)$$

$$M\{\tan^{-1} x\} = \frac{-\pi}{2p\, \cos(\frac{\pi p}{2})} = \tilde{g}(p)$$

and $\tilde{g}(p-1) = \dfrac{-\pi}{2(p-1)\cos(\frac{\pi p}{2} - \frac{\pi}{2})} = \dfrac{-\pi}{2(p-1)\sin\frac{\pi p}{2}}$

So $M\{f(x)\} = \dfrac{\pi}{2} \csc \dfrac{\pi p}{2}$

Q.7 (10 points) Show the Hankel transform

$$\mathcal{H}_o\{(a^2-r^2)H(a-r)\} = \frac{4a}{\alpha^3}J_1(a\alpha) - \frac{2a^2}{\alpha^2}J_0(a\alpha)$$

Sol: $\mathcal{H}_0\{(a^2-r^2)H(a-r)\} = \int_0^\infty r(a^2-r^2)H(a-r)J_0(\alpha r)\,dr$

$$= a^2\int_0^r J_0(\alpha r)\,dr - \int_0^r r\, r^2 J_0(\alpha r)\,dr$$

$$I_1 = a^2\int_0^a r J_0(\alpha r)\,dr = a^2\int_0 \frac{t}{\alpha}J_1(t)\frac{dt}{\alpha} \qquad \alpha r = t$$

$$= \frac{a^2}{\alpha^2}\int^{a\alpha} t J_1(t)\,dt = \frac{a^2}{\alpha^2}\cdot a\alpha\, J_1(a\alpha) = \frac{a^3}{\alpha}J_1(a\alpha)$$

$$I_2 = \int_0^{a\alpha}\frac{1}{\alpha^4}t^2\, t\, J_0(t)\,dt = \frac{1}{\alpha^4}\int_0^{a\alpha} t^2\frac{d}{dt}[tJ_1(t)]\,dt$$

$$= \frac{1}{\alpha^4}\left(t^2 t J_1(t)\Big|_0^{a\alpha} - 2\int_0^{a\alpha} t\cdot t J_1(t)\,dt\right)$$

$$= \frac{a^3}{\alpha}J_1(a\alpha) - \frac{2}{\alpha^4}\int_0^{a\alpha}\frac{d}{dt}[t^2J_2(t)]\,dt$$

$$= \frac{a^3}{\alpha}J_1(a\alpha) - \frac{2a^2}{\alpha^2}J_2(a\alpha)$$

$$\mathcal{H}_0\{(a^2-r^2)H(a-r)\} = \frac{2a^2}{\alpha^2}J_2(a\alpha) = \frac{2a}{\alpha^3}\,a\alpha\, J_2(a\alpha)$$

Now use $2\nu J_\nu(x) = x J_{\nu+1}(x) + x J_{\nu-1}(x)$

$\nu = 1$, $\quad xJ_2(x) = 2J_1(x) - xJ_0(x)$

So $\mathcal{H}_0\{(a^2-r^2)H(a-r)\} = \frac{2a}{\alpha^3}\left[2J_1(a\alpha) - a\alpha J_0(a\alpha)\right]$

$$= \frac{4a}{\alpha^3}J_1(a\alpha) - \frac{2a^2}{\alpha^2}J_0(a\alpha)$$

Q.8 (10 points) Find a solution $\Phi(x, y)$ of the Laplace equation

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0$$

under the condition $\lim\limits_{y \to 0+} = \begin{cases} T_o & x < -1 \\ T_1 & |x| < 1 \\ T_2 & x > 1 \end{cases}$. Hint: Use Poisson's formula.

Sol: We need find a Harmonic function in the upper half plane. Using Poisson's formula,

$$\Phi(x, y) = \frac{1}{\pi} \int \frac{y\, G(\eta)}{(x-\eta)^2 + y^2}\, d\eta$$

$$= \frac{1}{\pi} \int_{-\infty}^{-1} T_0 \frac{y}{(x-\eta)^2 + y^2}\, d\eta + \frac{1}{\pi} \int_{-1}^{1} T_1 \frac{y}{(x-\eta)^2 + y^2}\, d\eta$$

$$+ \frac{1}{\pi} \int_{1}^{\infty} T_2 \frac{y}{(x-\eta)^2 + y^2}\, d\eta$$

$$= -\frac{T_0}{\pi} \tan^{-1} \frac{x-\eta}{y} \Big|_{-\infty}^{-1} - \frac{T_1}{\pi} \tan^{-1} \frac{x-\eta}{y} \Big|_{-1}^{1}$$

$$- \frac{T_2}{\pi} \tan^{-1} \frac{x-\eta}{y} \Big|_{1}^{\infty}$$

$$= -\frac{T_0}{\pi} \left[\tan^{-1} \frac{x+1}{y} - \frac{\pi}{2} \right] - \frac{T_1}{\pi} \left[\tan^{-1} \frac{x-1}{y} - \tan^{-1} \frac{x+1}{y} \right]$$

$$- \frac{T_2}{\pi} \left[-\frac{\pi}{2} - \tan^{-1} \frac{x-1}{y} \right]$$

$$= \frac{T_0}{\pi} \tan^{-1} \frac{y}{x+1} - \frac{T_1}{\pi} \left[\tan^{-1} \frac{x-1}{y} - \frac{\pi}{2} - \tan^{-1} \frac{x+1}{y} + \frac{\pi}{2} \right]$$

$x+1 \quad \pi \quad 1 \qquad y \quad 2 \qquad y \quad 2\,)$

$$+\frac{T_2}{\pi}\left[\frac{\pi}{2}+\tan^{-1}\frac{x-1}{y}+\pi-\pi\right]$$

$$=\frac{T_0}{\pi}\tan^{-1}\frac{y}{x+1}+\frac{T_1}{\pi}\tan^{-1}\frac{y}{x-1}-\frac{T_1}{\pi}\tan^{-1}\frac{y}{x+1}$$

$$+T_2-\frac{T_2}{\pi}\tan^{-1}\frac{y}{x-1}$$

$$=\frac{T_0-T_1}{\pi}\tan^{-1}\frac{y}{x+1}+\frac{T_1-T_2}{\pi}\tan^{-1}\frac{y}{x-1}+T_2$$

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