PhD Comprehensive Exam

Duration: 180 minutes

ID:	
NAME:	

- Justify your answers thoroughly. For any theorem that you wish to cite, you should give its name and its statement.

Problem	Score
1	
2	
3	
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6	
8	
Total	/100

Problem 1

Show that a set E is measurable if and only if for every $\varepsilon > 0$, there exists a closed set F and open set $\mathcal{O}$ for which $F \subseteq E \subseteq \mathcal{O}$ and $m^*(\mathcal{O} \sim F) \leq \varepsilon$.

Solution: Suppose E is a measurable set and let $\varepsilon > 0$. There exists an open set $\mathcal{O}$ containing E and a closed set F contained in E for which $m^*(\mathcal{O} \sim E) < \varepsilon/2$ and $m^*(E \sim F) < \varepsilon/2$. By the measurability of E, we have

$$
\begin{aligned}
m^*(\mathcal{O} \sim F) &= m^*(\mathcal{O} \cap F^C \cap E) + m^*(\mathcal{O} \cap F^C \cap E^C) \\
&= m^*(E \sim F) + m^*(\mathcal{O} \sim E) \\
&< \varepsilon.
\end{aligned}
$$

Now fix $\varepsilon > 0$ and suppose that there is a closed set F and an open set O for which $F \subseteq E \subseteq \mathcal{O}$ and $m^*(\mathcal{O} \sim F) \leq \varepsilon$. Then

$$m^*(\mathcal{O} \sim E) \leq m^*(\mathcal{O} \sim F) < \varepsilon.$$

Since such sets can be found for all ε, E is measurable.

Problem 2

Assume that E is a set of finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E a.e. to a real-valued function f that is finite a.e. Show that the conclusion of Egoroff's Theorem still holds.

Solution: Assume E has finite measure and suppose $\{f_n\}$ is a sequence of measurable functions on E that converges pointwise a.e. to a function f that is finite a.e. Define

$$E_0 = \left\{x \in E : \lim_{n\to\infty} f_n(x) = f(x)\right\}$$

and $E_1 = \{x \in E : f(x) \text{ is finite}\}$. By assumption, $m(E \sim E_0) = m(E \sim E_1) = 0$.

Let $\tilde{E} = E_0 \cap E_1$ and observe that

$$0 \leq m(E \sim \tilde{E}) \leq m(E \sim E_1) + m(E \sim E_0) = 0.$$

Hence $m(E \sim \tilde{E}) = 0$.

Let $\varepsilon > 0$. By Egoroff's Theorem, there is a closed set F contained in $\tilde{E}$ for which

$$\{f_n\} \to f \text{ uniformly on } F \text{ and } m(\tilde{E} \sim F) < \varepsilon.$$

But then F is also a closed set contained in E that satisfies

$$\begin{aligned} m(E \sim F) &\leq m(\tilde{E} \sim F) + m(E \sim \tilde{E} \sim F) \\ &< \varepsilon + m(E \sim \tilde{E}) = \varepsilon. \end{aligned}$$

Problem 3

Consider two Lebesgue integrable functions f and g over $\mathbb{R}$ and two sequences of Lebesgue integrable functions $\{f_n\}$ and $\{g_n\}$ over $\mathbb{R}$. Assume that

(i) $\{f_n\}$ converges to f pointwise a.e. on $\mathbb{R}$,

(ii) $\{g_n\}$ converges to g pointwise a.e. on $\mathbb{R}$,

(iii) $|f_n| \leq g_n$ a.e. on $\mathbb{R}$ and

(iv) $\int_{\mathbb{R}} g_n$ converges to $\int_{\mathbb{R}} g$.

Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f$$

Solution: We need to show that

$$\lim \left(\int f_n - \int f \right) = 0.$$

In other words, since

$$\left| \int f_n - \int f \right| = \left| \int (f_n - f) \right| \leq \int |f_n - f|,$$

we need to show

$$\lim \int |f_n - f| = 0$$

First we have,

$$\begin{aligned} \int g + \int |f| &= \int (g + |f|) \\ &= \int \lim(g_n + |f|) \\ &= \int \lim(g_n + |f| - |f_n - f|) \end{aligned}$$

Note that $\lim |f_n - f| = 0$ and $|f_n - f| \leq |f_n| + |f| \leq g_n + |f|$. Hence $g_n + |f| - |f_n - f| \geq 0$. So we may proceed as

$$\begin{aligned} \int g + \int |f| &= \int \lim(g_n + |f| - |f_n - f|) \\ &\leq \liminf \int (g_n + |f| - |f_n - f|) \quad \text{(by Fatou's Lemma)} \\ &\leq \int g + \int |f| - \limsup \int |f_n - f|). \end{aligned}$$

Therefore $\limsup \int |f_n - f|) = 0$ and we have

$$\lim \int |f_n - f| = \limsup \int |f_n - f| = 0$$

Problem 4

Consider the sequence of functions

$$f_n(x) = \left(1 + \frac{x}{n}\right)^n.$$

(a) For all $n \geq 1$ and $x > -n$, show that f_n is monotone increasing and that

$$f_n(x) \leq e^x.$$

(b) Evaluate

$$\lim_{n\to\infty} \int_0^n f_n(x) e^{-3x} dx.$$

Solution: (a) Note that

$$\left(1+\frac{x}{n}\right)^n \leq e^x \quad \Leftrightarrow \quad n \ln\left(1+\frac{x}{n}\right) \leq x \quad \Leftrightarrow \quad \ln\left(1+\frac{x}{n}\right) = \ln\left(\frac{x+n}{n}\right) \leq \frac{x}{n} = \frac{x+n}{n} - 1$$

The statement is true since $\left(1+\frac{x}{n}\right) = \left(\frac{x+n}{n}\right) > 0$ and $\ln t \leq t - 1$ for all $t > 0$. To show that $\{f_n\}$ is monotone increasing, we treat n as a continuous variable and use L'Hospital rule

$$\frac{df_n}{dn} = f_n \left[\ln\left(1+\frac{x}{n}\right) - \frac{x}{x+n}\right] \geq 0 \quad \Leftrightarrow \quad \left[\ln\left(1+\frac{x}{n}\right) - \frac{x}{x+n}\right] \geq 0 \quad \text{since } f_n > 0.$$

So we need

$$\ln\left(\frac{x+n}{n}\right) \geq \frac{x}{x+n} \quad \Leftrightarrow \quad \ln\left(\frac{n}{x+n}\right) \leq -\frac{x+n}{x} = \frac{n}{x+n} - 1$$

which is true since $\frac{n}{x+n} > 0$ and $\ln t \leq t - 1$ for all $t > 0$.

(b) First we write

$$\lim_{n\to\infty} \int_0^n f_n(x) e^{-3x} dx = \lim_{n\to\infty} \int_0^\infty f_n(x) e^{-3x} \chi_{[0,n]} dx.$$

By part (a), the sequence $f_n(x)e^{-3x}\chi_{[0,n]}$ is nonnegative monotone increasing sequence and

$$f_n(x) e^{-3x} \chi_{[0,n]} \to e^x e^{-3x} = e^{-2x} \text{ pointwise }.$$

So by the Monotone Convergence Theorem we get

$$\lim_{n\to\infty} \int_0^n f_n(x) e^{-3x} dx = \lim_{n\to\infty} \int_0^\infty f_n(x) e^{-3x} \chi_{[0,n]} dx = \int_0^\infty e^{-2x} dx = \frac{1}{2}.$$

Problem 5

Let the function f be absolutely continuous on $[a,b]$. Show that f is Lipschitz on $[a,b]$ if and only if there exists $M > 0$ such that $f'(x) \leq M$ a.e. on $[a,b]$.

Solution: Suppose f is Lipschitz. Then there exists $M > 0$ such that

$$\left| \frac{f(x+t) - f(x)}{t} \right| \leq M$$

for any $x \in (a,b)$ and $t \neq 0$ that satisfy $x + t \in (a,b)$. Since f is absolutely continuous on $[a,b]$ then f is differentiable a.e. on (a,b). Hence

$$|f'(x)| = \lim_{t \to 0} \left| \frac{f(x+t) - f(x)}{t} \right| \leq M \quad \text{a.e.}$$

Conversely, suppose there exists $M > 0$ such that $|f'| \leq M$ a.e. on $[a,b]$. If x, x' are in $[a,b]$ and $x \leq x'$, then f is absolutely continuous on $[x,x']$. Therefore,

$$\begin{aligned} |f(x) - f(x')| &= \left| \int_x^{x'} f' \right| \\ &\leq \int_x^{x'} |f'| \\ &\leq M(x' - x), \end{aligned}$$

that is, f is Lipschitz on $[a,b]$.

Problem 6

Let E be a measurable set and $1 \leq p < \infty$ and $f_n \to f$ in $L^p(E)$.

(a) If the Lebesgue measure, m, of E is finite that is $m(E) < \infty$, show that $f_n \to f$ in $L^s(E)$ for some $1 \leq s < p$.

(b) If $f_n \to f$ a.e on E and there exists a real number M such that $|f_n| \leq M$ a.e. for all n, show that $f_n \to f$ in $L^r(E)$ for some r such that $1 \leq p < r < \infty$.

Solution:

(a) Since $s < p$, let $p_1 = \frac{p}{s}$ and q_1 be such that $\frac{1}{p_1} + \frac{1}{q_1} = 1$, then using Holder Inequality, we have

$$\begin{aligned}
\int_E |f_n - f|^s dm &\leq \| |f_n - f|^s \|_{p1} \|1\|_{q_1} \\
&= \left(\int_E (|f_n - f|^s)^{p_1} dm \right)^{1/p_1} \left(\int_E dm \right)^{1/q_1} \\
&= \left(\int_E |f_n - f|^p dm \right)^{s/p} (m(E))^{1/q_1} \\
&= (\|f_n - f\|_p)^s \, (m(E))^{1/q_1}
\end{aligned}$$

Since $m(E) < \infty$ and $\|f_n - f\|_p \to 0$ as $n \to \infty$, then $f_n \to f$ in $L^s(E)$.

(b) Let $r > p$. Since $f_n \to f$ a.e. on E and $|f_n| \leq M$ then $|f| \leq M$. Therefore $|f_n - f| \leq 2M$. Now we have

$$\begin{aligned}
(\|f_n - f\|_r)^r &= \int_E |f_n - f|^r dm \\
&= \int_E |f_n - f|^{r-p} |f_n - f|^p dm \\
&\leq \int_E (2M)^{r-p} |f_n - f|^p dm \\
&= (2M)^{r-p} \int_E |f_n - f|^p dm \\
&= (2M)^{r-p} (\|f_n - f\|_p)^p \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

Hence $f_n \to f$ in $L^r(E)$

Problem 7

Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{h_n\}$ be a sequence of nonnegative integrable functions on X. Suppose that $\{h_n(x)\} \to 0$ for almost all $x \in X$. Show that

$$\lim_{n\to\infty} \int_X h_n \, d\mu = 0$$

if and only if $\{h_n\}$ is uniformly integrable and tight over X.

Solution: If $\{h_n\}$ is uniformly integrable and tight over X, then by the Vitaly Convergence Theorem

$$\lim_{n\to\infty} \int_X h_n \, d\mu = 0.$$

Conversely, suppose $\lim_{n\to\infty} \int_X h_n \, d\mu = 0$. There exists a natural number N such that

$$\int_X h_n \, d\mu < \varepsilon$$

for all $n \geq N$. We know that the finite collection of functions $\{h_n\}_{n=1}^{N}$ is tight over X. We therefore can find a set of finite measure $X_0 \in X$ such that

$$\int_{X \sim X_0} |h_n| \, d\mu < \varepsilon.$$

for all $n < N$. Since

$$\int_{X \sim X_0} |h_n| \, d\mu \leq \int_X |h_n| \, d\mu < \varepsilon.$$

for all $n \geq N$. We conclude that $\{h_n\}$ is tight over X.
Note that the finite collection of functions $\{h_n\}_{n=1}^{N}$ is uniformly integrable.
Now Choose $\delta > 0$ such that for $n \geq N$, if $A \subseteq X$ is measurable and $m(A) < \delta$ then

$$\int_A |h_n| d\mu < \varepsilon.$$

Hence we have that
if $A \subseteq X$ is measurable and $m(A) < \delta$, then $\int_A |h_n| d\mu < \varepsilon$ for all n. Therefore $\{h_n\}$ is uniformly integrable.

Problem 8

Let $X = Y$ be the interval $[0,1]$ with $\mathcal{A} = \mathcal{B}$ the class of Borel sets. Let $\mu = \nu$ be the Lebesgue measure. Consider the function f on $X \times Y$ defined as

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}, \quad (x,y) \in X \times Y.$$

(a) Show that

$$\int_X \int_Y f d\mu d\nu \neq \int_Y \int_X f d\nu d\mu.$$

(b) Does part (a) contradict Fubini's theorem? why?

Solution:

(a) First, we compute

$$\begin{aligned}
\int_X \int_Y f d\mu d\nu &= \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} dx dy = \int_0^1 \left[\int_0^1 \frac{2x^2}{(x^2+y^2)^2} dx - \int_0^1 \frac{x^2+y^2}{(x^2+y^2)^2} dx \right] dy \\
&= \int_0^1 \left[\int_0^1 -x \frac{-2x}{(x^2+y^2)^2} dx - \int_0^1 \frac{dx}{x^2+y^2} \right] dy \\
&= \int_0^1 \left[\int_0^1 -x \frac{d}{dx} \left(\frac{1}{x^2+y^2} \right) - \int_0^1 \frac{dx}{x^2+y^2} \right] dy \\
&= \int_0^1 \left[\frac{-x}{x^2+y^2} \bigg|_{x=0}^1 + \int_0^1 \frac{dx}{x^2+y^2} - \int_0^1 \frac{dx}{x^2+y^2} \right] dy \\
&= \int_0^1 \frac{-1}{1+y^2} dy = -\tan^{-1} y \bigg|_0^1 = -\frac{\pi}{4}.
\end{aligned}$$

Then

$$\int_Y \int_X f d\mu d\nu = \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} dy dx = \frac{\pi}{4}.$$

(b) Part (a) does not contradict Fubini's theorem, since f is not integrable over $X \times Y$ with respect to the product measure $\mu \times \nu$. Indeed

$$\begin{aligned}
\int_{X \times Y} |f| d(\mu \times \nu) &= \int_{[0,1]\times[0,1]} \left| \frac{x^2 - y^2}{(x^2+y^2)^2} \right| d(\mu \times \nu) \\
&= \int_0^1 \int_0^1 \left| \frac{x^2 - y^2}{(x^2+y^2)^2} \right| dx dy \\
&= \int_0^1 \left[\int_0^y \frac{y^2 - x^2}{(x^2+y^2)^2} dx + \int_y^1 \frac{x^2 - y^2}{(x^2+y^2)^2} dx \right] dy \\
&= \int_0^1 \left[\frac{x}{x^2+y^2} \bigg|_0^y - \frac{x}{x^2+y^2} \bigg|_y^1 \right] dy \quad \text{(from part (a) solution)} \\
&= \int_0^1 \left[\frac{1}{2y} - \frac{1}{1+y^2} + \frac{1}{2y} \right] dy \\
&= \int_0^1 \left[\frac{1}{y} - \frac{1}{1+y^2} \right] dy \\
&= \lim_{t \to 0+} \int_t^1 \frac{dy}{y} - \tan^{-1}(y) \bigg|_0^1 \\
&= \lim_{t \to 0+} -\ln t - \frac{\pi}{4} = \infty
\end{aligned}$$