PhD Comprehensive Exam

Duration: 180 minutes

ID:	
NAME:	

- Justify your answers thoroughly. For any theorem that you wish to cite, you should give its name and its statement.

Problem	Score
1	
2	
3	
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8	
Total	/100

Problem 1

Let E have finite outer measure. Show that E is measurable if and only if for each open, bounded interval (a, b), $b - a = m^*((a, b) \cap E) + m^*((a, b) \sim E)$.

Solution: Let E be measurable. We have

$$\begin{aligned} b - a = m^*((a, b)) &= m^*\left((a, b) \cap E\right) + m^*\left((a, b) \cap E^C\right) \\ &= m^*((a, b) \cap E) + m^*\left((a, b) \sim E\right) \end{aligned}$$

Conversely, suppose the equality holds. Let A be any set with $m^*(A) < \infty$.
Let $\varepsilon > 0$. Choose a countable collection of open intervals $\{(a_k, b_k\}_{k=1}^{\infty}$ that covers A and

$$\sum_{k=1}^{\infty} \ell(a_k, b_k) < m^*(A) + \varepsilon.$$

Thus,

$$\begin{aligned} m^*(A) &> \sum_{k=1}^{\infty} (b_k - a_k) - \varepsilon \\ &\geq \sum_{k=1}^{\infty} m^*((a_k, b_k) \cap E) + m^*((a, b) \sim E^C) - \varepsilon \\ &\geq \bigcup_{k=1}^{\infty} m^*\left[m^*(a_k, b_k) \cap E\right] + \bigcup_{k=1}^{\infty} m^*\left[m^*(a_k, b_k) \sim E\right] - \varepsilon \\ &\geq m^*(A \cap E) + m^*(A \sim E^C) - \varepsilon. \end{aligned}$$

Since this holds for every $\varepsilon > 0$, we obtain

$$m^*(A) \geq m^*(A \cap E) + m^*\left(A \sim E^C\right).$$

Hence A is measurable.

Problem 2

(a) For the function f and the set F in the statement of Lusin's Theorem, show that the restriction of f to F is a continuous function.

(b) Prove the extension of Lusin's Theorem to the case that f is not necessarily real-valued, but may be finite a.e.

Solution:

1. Fix $x \in F$ and let $\varepsilon > 0$. Since g is continuous, there exists $\delta > 0$ such that

$$\text{if } |x' - x| < \delta, \text{ then } |g(x') - g(x)| < \varepsilon.$$

Since $f = g$ on F, we have

$$\text{if } x \in F \text{ and } |x' - x| < \delta, \text{ then } |f(x') - f(x)| < \varepsilon.$$

Thus, f is continuous on F.

2. Suppose f is an extended real-valued function on E that is finite a.e. Let $\varepsilon > 0$ and let

$$E_0 = \{x \in E : \text{f is finite}\}.$$

Lusin's theorem implies that there exists a continuous function g on $\mathbb{R}$ and a closed set F contained in E_0 for which

$$f = g \text{ on } F \text{ and } m(E_0 \sim F) < \varepsilon.$$

But since

$$E = (E \sim E_0) \cup E_0 \text{ and } m(E \sim E_0) = 0,$$

we also have

$$m(E \sim F) \leq m(E \sim E_0 \sim F) + m(E_0 \sim F) = m(E_0 \sim F) < \varepsilon.$$

Problem 3

Let $\{f_n\}$ be a sequence of integrable functions on E for which $f_n \to f$ a.e. on E and f is integrable over E. Show that

$$\int_E |f - f_n| \to 0$$

if and only if $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$. (Hint: Use the General Lebesgue Dominated Convergence Theorem.)

Solution: Suppose $\int_E |f - f_n| \to 0$. Since

$$||f_n| - |f|| \leq |f_n - f|$$

on E for all n and $|f_n| - |f| \to 0$ a.e. on E, we have

$$\lim_{n\to\infty} \int_E (|f_n| - |f|) = 0$$

by the General Lebesgue Dominated Convergence Theorem. Since f is integrable, we have

$$\lim_{n\to\infty} \int_E |f_n| = \int_E |f|.$$

Conversely, suppose

$$\lim_{n\to\infty} \int_E |f_n| = \int_E |f|.$$

Then

$$\lim_{n\to\infty} \int_E (|f_n| + |f|) = 2\int_E |f| < \infty.$$

Since

$$|f_n - f| \leq |f_n| + |f|$$

on E for all n and $|f_n - f| \to 0$ a.e. on E, we have

$$\int_E |f - f_n| \to 0.$$

by the General Lebesgue Dominated Convergence Theorem.

Problem 4

Consider the sequence of functions

$$f_n(x) = \left(1 + \frac{x}{n}\right)^n.$$

(a) For all $n \geq 1$ and $x > -n$, show that f_n is monotone increasing and that

$$f_n(x) \leq e^x.$$

(b) Evaluate

$$\lim_{n\to\infty} \int_0^n f_n(x) e^{-5x} dx.$$

Solution: (a) Note that

$$\left(1+\frac{x}{n}\right)^n \leq e^x \quad \Leftrightarrow \quad n\ln\left(1+\frac{x}{n}\right) \leq x \quad \Leftrightarrow \quad \ln\left(1+\frac{x}{n}\right) = \ln\left(\frac{x+n}{n}\right) \leq \frac{x}{n} = \frac{x+n}{n} - 1$$

The statement is true since $\left(1+\frac{x}{n}\right) = \left(\frac{x+n}{n}\right) > 0$ and $\ln t \leq t-1$ for all $t > 0$. To show that $\{f_n\}$ is monotone increasing, we treat n as a continuous variable and use L'Hospital rule

$$\frac{df_n}{dn} = f_n\left[\ln\left(1+\frac{x}{n}\right) - \frac{x}{x+n}\right] \geq 0 \quad \Leftrightarrow \quad \left[\ln\left(1+\frac{x}{n}\right) - \frac{x}{x+n}\right] \geq 0 \quad \text{since } f_n > 0.$$

So we need

$$\ln\left(\frac{x+n}{n}\right) \geq \frac{x}{x+n} \quad \Leftrightarrow \quad \ln\left(\frac{n}{x+n}\right) \leq -\frac{x+n}{x} = \frac{n}{x+n} - 1$$

which is true since $\frac{n}{x+n} > 0$ and $\ln t \leq t - 1$ for all $t > 0$.

(b) First we write

$$\lim_{n\to\infty} \int_0^n f_n(x) e^{-5x} dx = \lim_{n\to\infty} \int_0^\infty f_n(x) e^{-5x} \chi_{[0,n]} dx.$$

By part (a), the sequence $f_n(x)e^{-5x}\chi_{[0,n]}$ is nonnegative monotone increasing sequence and

$$f_n(x)e^{-5x}\chi_{[0,n]} \to e^x e^{-5x} = e^{-4x} \text{ pointwise .}$$

So by the Monotone Convergence Theorem we get

$$\lim_{n\to\infty} \int_0^n f_n(x) e^{-5x} dx = \lim_{n\to\infty} \int_0^\infty f_n(x) e^{-5x} \chi_{[0,n]} dx = \int_0^\infty e^{-4x} dx = \frac{1}{4}.$$

Problem 5

If f is continuous on $[a,b]$ and f' exists and is bounded on (a,b), then show that f is absolutely continuous on $[a,b]$.

Solution: Since f' exists and bounded on (a,b), there exists $M > 0$ such that $f'(x) < M$ for all $x \in (a,b)$. Let $\varepsilon > 0$ and coinsider

$$\sum_{i=1}^{n} |f(d_i) - f(c_i)|$$

where $\{[c_i, d_i] : 1 \le i \le n\}$ is a finite collection of non-overlapping intervals in $[a,b]$ such that

$$\sum_{i=1}^{n} |d_i - c_i| < \varepsilon/M.$$

We have

$$\sum_{i=1}^{n} |f(d_i) - f(c_i)| = \sum_{i=1}^{n} \frac{|f(d_i) - f(c_i)|}{|d_i - c_i|} |d_i - c_i|.$$

The Mean Value Theorem tells us that for every i there exists $x_i \in [c_i, d_i]$ such that

$$\frac{|f(d_i) - f(c_i)|}{|d_i - c_i|} = f'(x_i) < M.$$

Thus, we can write

$$\begin{aligned}
\sum_{i=1}^{n} |f(d_i) - f(c_i)| &= \sum_{i=1}^{n} \frac{|f(d_i) - f(c_i)|}{|d_i - c_i|} |d_i - c_i| \\
&< \sum_{i=1}^{n} M |d_i - c_i| \\
&= M \sum_{i=1}^{n} |d_i - c_i| \\
&< M \frac{\varepsilon}{M} = \varepsilon.
\end{aligned}$$

Hence, f is absolutely continuous on $[a,b]$.

Problem 6

Assume E has finite measure and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence of measurable functions that converge pointwise a.e. on E to f. For $1 \leq p < \infty$, show that $\{f_n\} \to f$ in $L^p(E)$ if there is $\alpha > 0$ such that $\{f_n\}$ belongs to and is bounded as a subset of $L^{p+\alpha}(E)$.

Solution: By the assumption, there exists a real number M such that $\|f_n\|_{p+\alpha} \leq M$ for all n. By Fatou's Lemma we have

$$\int_E |f|^{p+\alpha} \leq \liminf_{n\to\infty} \int_E |f_n|^{p+\alpha} \leq M^{p+\alpha} < \infty.$$

That is $f \in L^{p+\alpha}(E)$. This implies that f and $\{f_n\}$ are in $L^p(E)$. It remains to show that

$$\|f_n - f\| \to 0.$$

By Holder's Inequaltiy, we have

$$\int_A |f_n - f|^p \leq \|f_n - f\|_{p+\alpha}^p \cdot m(A)^{\alpha/(p+\alpha)}$$

for any measurable set $A \subseteq E$. By Minkowski's Inequality,

$$\|f_n - f\|_{p+\alpha} \leq \|f_n\|_{p+\alpha} + \|f\|_{p+\alpha} \leq 2 \cdot M.$$

Hence,

$$\int_A |f_n - f|^p \leq 2^p \cdot M^p \cdot m(A)^{\alpha/(p+\alpha)}.$$

Thus, for any $\varepsilon > 0$, we have

$$\int_A |f_n - f|^p < \varepsilon$$

as long as

$$m(A) < \left(\frac{\varepsilon}{2^p \cdot M^p}\right)^{(p+\alpha)/\alpha}.$$

Therefore $\{|f_n - f|^p\}$ is uniformly integrable over E, which implies that

$$\lim_{n\to\infty} \int_E |f_n - f|^p = 0$$

by the Vitali Convergence Theorem.

Problem 7

Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{h_n\}$ be a sequence of nonnegative integrable functions on X. Suppose that $\{h_n(x)\} \to 0$ for almost all $x \in X$. Show that

$$\lim_{n\to\infty} \int_X h_n \, d\mu = 0$$

if and only if $\{h_n\}$ is uniformly integrable and tight over X.

Solution: If $\{h_n\}$ is uniformly integrable and tight over X, then by the Vitaly Convergence Theorem

$$\lim_{n\to\infty} \int_X h_n \, d\mu = 0.$$

Conversely, suppose $\lim_{n\to\infty} \int_X h_n \, d\mu = 0$. There exists a natural number N such that

$$\int_X h_n \, d\mu < \varepsilon$$

for all $n \geq N$. We know that the finite collection of functions $\{h_n\}_{n=1}^{N}$ is tight over X. We therefore can find a set of finite measure $X_0 \in X$ such that

$$\int_{X \sim X_0} |h_n| \, d\mu < \varepsilon.$$

for all $n < N$. Since

$$\int_{X \sim X_0} |h_n| \, d\mu \leq \int_X |h_n| \, d\mu < \varepsilon.$$

for all $n \geq N$. We conclude that $\{h_n\}$ is tight over X.
Note that the finite collection of functions $\{h_n\}_{n=1}^{N}$ is uniformly integrable.
Now Choose $\delta > 0$ such that for $n \geq N$, if $A \subseteq X$ is measurable and $m(A) < \delta$ then

$$\int_A |h_n| d\mu < \varepsilon.$$

Hence we have that

if $A \subseteq X$ is measurable and $m(A) < \delta$, then $\int_A |h_n| d\mu < \varepsilon$ for all n.

Therefore $\{h_n\}$ is uniformly integrable.

Problem 8

Let $\mathbb{N}$ be the set of natural numbers, and let $\mathcal{M} = 2^{\mathbb{N}}$ and c be the counting measure by setting $c(E)$ equal to the number of points in E if E is finite and ∞ if E is infinite. Let $(X, \mathcal{A}, \mu) = (Y, \mathcal{B}, \nu) = (\mathbb{N}, \mathcal{M}, c)$. Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 2 - 2^{-x} & \text{if} \quad x = y, \\ -2 + 2^{-x} & \text{if} \quad x = y + 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that f is measurable with respect to the product measure $c \times c$.

(b) Show that

$$\int_{\mathbb{N}} \left[\int_{\mathbb{N}} f(m,n) dc(m) \right] dc(n) \neq \int_{\mathbb{N}} \left[\int_{\mathbb{N}} f(m,n) dc(n) \right] dc(m).$$

(c) Is this a contradiction either of Fubini's theorem or Tonelli's theorem ?

Solution:

(a) Since $\mathbb{N}$ is countable, it has measure zero. The only set which has $c \times c$ measure zero is the empty set. Hence, $c \times c$ measurable set of $\mathbb{N} \times \mathbb{N}$ are all the subsets. Therefore, f is measurable.

(b) We know that

$$\int_{\mathbb{N}} f dc = \sum_{k=1}^{\infty} f(k).$$

For convenience, let $dx = dc(m)$ and $dy = dc(n)$ Hence,

$$\begin{aligned} \int_{\mathbb{N}} \left(\int_{\mathbb{N}} f(x,y) dx \right) dy &= \int_{\mathbb{N}} \left(\sum_{x \in \mathbb{N}} f(x,y) \right) dy \\ &= \int_{\mathbb{N}} \left[f(y,y) + f(y+1,y) \right] dy \\ &= \int_{\mathbb{N}} \left(2 - 2^{-y} - 2 + 2^{-y-1} \right) dy = \int_{\mathbb{N}} \frac{-1}{2^{y+1}} dy \\ &= \sum_{y=1}^{\infty} \frac{-1}{2^{y+1}} = -\frac{1}{2} \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{\mathbb{N}} \left(\int_{\mathbb{N}} f(x,y) dy \right) dx &= \int_{\mathbb{N}} \left(\sum_{y \in \mathbb{N}} f(x,y) \right) dx, \quad \text{let } h(x) = \sum_{y \in \mathbb{N}} f(x,y) \\ &= \int_{\mathbb{N}} [h(x)] \, dx \\ &= \sum_{x=1}^{\infty} h(x) = h(1) + \sum_{x=2}^{\infty} h(x) = h(1) \\ &= f(1,1) = 1.5 \end{aligned}$$

(c) No, because f assumes negative values and not integrable.