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Department of Mathematics

Comprehensive Exam- Math 533
Complex Analysis
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Solution

1. Find all entire functions f such that

$$|f(z)| \leq |z|^{2/3}, \text{ for all } z \in \mathbb{C}.$$

Solution: As f is an entire function, we can write $f(z) = \sum_{n\geq 0} a_n z^n$, with

$$a_n = \frac{1}{2\pi i}\int_{|z|=R} \frac{f(z)}{z^{n+1}}\,dz. \quad \text{for all } R > 0.$$

By assumption, we have

$$|a_n| \leq \frac{1}{R^{n-2/3}}.$$

By taking the limit as $R \to \infty$, we deduce that

$$a_n = 0 \quad \forall n \geq 1,$$

and f is constant. Since $f(0) = 0$, we conclude that

$$f = 0.$$

2. (a) (5 points) Let f be an entire function such that $|\text{Im } f(z)| < \pi$ for all $z \in \mathbb{C}$. Show that f is constant.

   (b) (5 points) Show that there is no nonconstant entire function g such that $g(\mathbb{C}) \subset \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$.

   (c) (10 points)

      (i) Show that $T(z) = \dfrac{z-1}{z+1}$ maps $\mathbb{C} \setminus [-1, 1]$ onto $\mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$.

      (ii) Show that if h is an entire function such that $h(\mathbb{C}) \subset \mathbb{C} \setminus [-1, 1]$, then h is a constant.

Solution:

(a) Consider the function
$$g(z) = e^{-if(z)}.$$
g is entire with
$$|g(z)| \leq e^{\text{Im} f(z)} \leq e^{\pi}.$$
By Liouville theorem, deduce that g is constant and so f is constant.

(b)
$$g : \mathbb{C} \to \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$$
Consider
$$h(z) = \text{Log } g(z).$$
h is an entire function with $|\text{Im } h(z)| < \pi$, we deduce that h is constant and so g is a constant too.

(c) (i) Easy to check.

(ii) Consider
$$k := T \circ h : \mathbb{C} \to \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}.$$
By (b) k is constant and we conclude that h is constant.

3. (10 points) Describe all entire functions f such that

$$|f(z)| \leq |\sin(z)| \text{ for all } z \in \mathbb{C}.$$

Solution: Let

$$g(z) = \frac{f(z)}{\sin z}.$$

g has singularities at $\pi\mathbb{Z}$. By assumption, we have

$$|g(z)| \leq 1 \quad \forall z \in \mathbb{C} \setminus \pi\mathbb{Z}.$$

We deduce that all the singularities are removable and g extends to an entire bounded function, so it is constant by Liouville theorem. Thus

$$f(z) = c \sin z, \text{ with } \quad |c| \leq 1.$$

4. Suppose that $f : \Delta \to \Delta$ is analytic on the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and continuous on $|z| = 1$ such that $f(0) = f(a) = 0$, where $a \in \Delta$, and $a \neq 0$.

   (a) (5 points) Show that $|f'(0)| \leq |a|$. (Hint: Consider $z\varphi_a(z)$, where $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$)

   (b) (5 points) Find f, if $|f'(0)| = |a|$.

Solution:

(a) Consider the function

$$g(z) = \frac{f(z)}{\varphi_a(z)}.$$

Check the following

(i) g is analytic on Δ,with $g(0) = 0$.

(ii) g sends Δ to Δ.

By Schwarz Lemma, we deduce that $|g'(0)| \leq 1$.

$$|g'(0)| = \lim_{z \to 0} |\frac{g(z)}{z}| = \frac{|f'(0)|}{|a|}.$$

(b) By Schwarz lemma, if $|g'(0)| = 1$, then $g(z) = e^{i\theta}z$, and

$$f(z) = e^{i\theta} z\varphi_a(z).$$

5. (a) (10 points) Let $\mathscr{C}$ be a simply closed, positively oriented contour and f be an analytic function inside and on $\mathscr{C}$. Assume that f does not vanish on $\mathscr{C}$. Let $\{a_1, \ldots, a_N\}$ be the zeros of f inside $\mathscr{C}$ (counted with multiplicities). Show that

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{zf'(z)}{f(z)}\,dz = a_1 + \ldots + a_N.$$

(b) (5 points) Application: Let $n \in \mathbb{N}$, $n \geq 2$. Find

$$\oint_{|z|=2} \frac{z^n}{z^n - 1}\,dz$$

Solution: (a) By assumption

$$f(z) = (z - a_1)\ldots(z - a_n)g(z),$$

where g is analytic and $g(z) \neq 0$, for all z inside $\mathscr{C}$. Hence

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{z}{z - a_1} + \ldots + \frac{z}{z - a_n}.$$

As

$$\int_{\mathscr{C}} \frac{zg'(z)}{g(z)}dz = 0, \quad \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{z}{z - a_i}dz = a_i, \text{for} i \in [\![1..N]\!].$$

we deduce that

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{zf'(z)}{f(z)}\,dz = a_1 + \ldots + a_N.$$

(b) Apply (a) to $f(z) = z^n - 1$, we get

$$\oint_{|z|=2} \frac{z^n}{z^n - 1}\,dz = \frac{2\pi i}{n} Z(f),$$

where $Z(f)$ is the sum of zeros of f inside $|z| = 2$. As the sum of nth roots of unity is zero, for $n \geq 2$, we conclude that

$$\oint_{|z|=2} \frac{z^n}{z^n - 1}\,dz = 0.$$

6. Let $n \in \mathbb{N}$, $n \geq 1$ and $\alpha \in (-1, 1)$, with $\alpha \neq 0$.

   (a) (10 points) Compute $\displaystyle\int_0^{2\pi} \frac{e^{in\theta}}{(e^{i\theta}-\alpha)(e^{-i\theta}-\alpha)}\,d\theta$

   (b) (5 points) Deduce $I = \displaystyle\int_0^{2\pi} \frac{\cos n\theta}{1-2\alpha\cos\theta+\alpha^2}\,d\theta$.

Solution: (a) Use $z = e^{i\theta}$ to deduce that

$$J := \int_0^{2\pi} \frac{e^{in\theta}}{(e^{i\theta}-\alpha)(e^{-i\theta}-\alpha)}\,d\theta = -i\oint_{|z|=1} f(z)dz,$$

where

$$f(z) = \frac{z^n}{(z-\alpha)(1-\alpha z)}.$$

Using the residue theorem, we deduce that

$$J = 2\pi \mathrm{Res}\,(f, \alpha) = \frac{2\pi\alpha^n}{1-\alpha^2}.$$

(b)

$$I = \mathrm{Re}\, J = \frac{2\pi\alpha^n}{1-\alpha^2}.$$

7. (15 points) Let $f(z) = z^6 - 2z + 4$. Show that all the zeros of f lie on the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

Solution: On $|z| = 2$, $|z^6| = 64$, and $|f(z) - z^6| \leq 2|z| + 4 \leq 8 < 64$, Using Rouché theorem, we deduce that the number of zeros of f inside $|z| = 2$ is 6. In addition the f has no zeros inside $|z| = 1$, since for $|z| \leq 1$, we have $|f(z)| \geq 4 - |z^6 - 2z| \geq 1$.