King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH533 - Complex Variables I
Comprehensive Exam – Term 212
Exam Solution

Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Exercise 1

Compute

$$\int_0^{2\pi} \frac{\cos\theta}{\cos\theta - i}\, d\theta$$

Solution: Let I denote

$$I = \int_0^{2\pi} \frac{\cos\theta}{\cos\theta - i}\, d\theta.$$

Put $z = e^{i\theta}$, so $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$ and $d\theta = \frac{dz}{iz}$. We obtain

$$I = \oint_{|z|=1} \frac{z^2+1}{iz(z^2 - 2iz + 1)}\, dz.$$

The function

$$f(z) = \frac{z^2+1}{iz(z^2 - 2iz + 1)}$$

has 3 simple poles at $z = 0$ and $z = (1 \pm \sqrt{2})i$. Remark that only 0 and $(1-\sqrt{2})i$ are inside the unit circle. By the residue theorem, we have

$$I = 2\pi i(\mathrm{Res}(f, 0) + \mathrm{Res}(f, (1-\sqrt{2})i).$$

Next, we prove that

$$(\mathrm{Res}(f, 0) = -i, \quad \mathrm{Res}(f, (1-\sqrt{2})i = \frac{1}{\sqrt{2}}i.$$

Finally,

$$I = 2\pi(1 - \frac{1}{\sqrt{2}}).$$

Exercise 2

Let f and g be polynomials with $\deg(g) > \deg(f) + 1$.

(a) Show that $\lim_{R\to\infty} \int_{|z|=R} \frac{f(z)}{g(z)}\, dz = 0$.

(b) Use (a) to show that the sum of the residues of $\frac{f}{g}$ at all its poles is zero.

Solution:

(a) Let $n = \deg f$ and $m = \deg g$, by assumption, we have $m \geq n + 2$. We have

$$\frac{f(z)}{g(z)} = z^{n-m} H(z),$$

where H is a rational function having finite limit as $|z|$ goes to infinity. Hence for $|z|$ large, it yields

$$\left| \frac{f(z)}{g(z)} \right| \leq M |z|^{n-m}.$$

Therefore, for R large,

$$\left| \int_{|z|=R} \frac{f(z)}{g(z)}\, dz \right| \leq \frac{2\pi M}{R^{m-n-1}} \to 0,$$

and we conclude $\lim_{R\to\infty} \int_{|z|=R} \frac{f(z)}{g(z)}\, dz = 0$.

(b) As f/g is a rational function, it has a finite number of poles. By the residue theorem, we get

$$\int_{|z|=R} \frac{f(z)}{g(z)}\, dz = \text{sum of the residues of } \frac{f}{g},$$

for any R sufficiently large. By (a), we conclude that sum of the residues of f/g is zero.

Exercise 3

Let f and g be two entire functions such that

$$|f(z)| \leq |g(z)| \text{ for all } z \in \mathbb{C}.$$

Show that $f = cg$, for some constant $c \in \mathbb{C}$ with $|c| \leq 1$.

Solution: First, we may assume that g is non zero. As if $g = 0$, the conclusion is trivial. Consider

$$h(z) = \frac{f(z)}{g(z)}.$$

The function h has isolated singularities. These singularities are removable as h is bounded by assumption. Therefore h can be extended to an entire function, denoted by $\tilde{h}$ satisfying $|\tilde{h}(z)| \leq 1$. By Liouville theorem, we conclude that $\tilde{h}$ is equal to some constant c with $|c| \leq 1$. Finally, we conclude that $f(z) = cg(z)$ with $|c| \leq 1$.

Exercise 4

Let $\mathbb{D}$ be the unit disc and $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im(z)} > 0\}$.

(a) Show that the function $\varphi : \mathbb{H} \to \mathbb{D}$ defined by $\varphi(z) = \dfrac{z-i}{z+i}$ is an analytic bijection with an analytic inverse.

(b) Let $f : \mathbb{D} \setminus \{0\} \to \mathbb{H}$ be an analytic function. Study the nature of its singularity at zero.

Solution:

(a) The Cayley transform $\varphi(z) = \dfrac{z-i}{z+i}$ sends $\{\infty, 1, -1\}$ to $\{1, -i, i\}$. φ maps the real line to the unit circle. Furthermore, since φ is continuous and i is taken to 0 by φ, the upper half-plane is mapped to the unit disc.

(b) Let $f : \mathbb{D} \setminus \{0\} \to \mathbb{H}$ be an analytic function. Then

$$g := \varphi \circ f : \mathbb{D} \setminus \{0\} \to \mathbb{D}$$

is analytic and bounded, thus g has a removable singularity at zero, and it follows that f has a removable singularity at zero.

Exercise 5

Let f be an analytic function on a nonempty open connected set $\Omega \subset \mathbb{C}$. Let $a \in \Omega$ be a local minimum of $|f|$.

(a) Prove that either $f(a) = 0$ or f is constant on Ω.

(b) Prove or disprove that there exists an analytic function f on the unit disc $\mathbb{D}$ such that $|f(z)|^2 = |z|^2 + 1$ for all $z \in \mathbb{D}$.

Solution:

(a) Suppose that $f(a) \neq 0$, by assumption, there exists $R > 0$, such that $D(a, R) \subset \Omega$ and

$$|f(a)| = \min_{z \in D(a,R)} |f(z)| > 0.$$

Thus $g := \dfrac{1}{f}$ is analytic on $D(a, R)$ and $|g|$ has a local maximum at a. Hence f is constant on Ω as Ω is connected.

(b) If there exists such function f such that $|f(z)|^2 = |z|^2 + 1$. Remark that $|f(z)| \geq 1$ and $|f(0)| = 1$. We deduce that $|f|$ attains its minimum at $z = 0$. By (a), it follows that f is constant, and we get a contradiction.

Exercise 6

Let (f_n) be a sequence of analytic functions inside and on $|z| = 1$. Suppose that f_n converges uniformly to f inside and on $|z| = 1$.
Show if f has no zeros on $|z| = 1$, then the number of zeros of f inside $|z| = 1$ is equal to the number of zeros of f_n inside $|z| = 1$ for sufficiently large n.

Solution: Let

$$\varepsilon = \min_{|z|=1} |f(z)| > 0.$$

As f_n converges uniformly to f on $|z| = 1$, for n large, we get

$$\sup_{|z|=1} |f_n(z) - f(z)| < \varepsilon,$$

which implies that

$$|f_n(z) - f(z)| < |f(z)| \text{ on } |z| = 1.$$

By Rouché theorem, we conclude that for n large, the number of zeros of f_n is equal to the number of zeros of f inside $|z| = 1$.

Exercise 7

Let $\Omega \subset \mathbb{C}$ be a *bounded domain* and let

$$f : \Omega \to \Omega$$

be an analytic function. Suppose that $f(z_0) = z_0$ for a point z_0 in Ω. Let

$$f_n := \underbrace{f \circ f \circ \cdots \circ f}_{n\text{-times}}$$

(a) Prove by induction that $(f_n)'(z_0) = (f'(z_0))^n$, for all $n \geq 1$.

(b) Prove that $|(f_n)'(z_0)| \leq C$ for all $n \geq 1$, for some constant C.

(c) Deduce that $|f'(z_0)| \leq 1$.

(d) In addition, assume that f is an automorphism of Ω. What is the value of $|f'(z_0)|$?

Solution:

(a) Use the chain rule

(b) Let $r > 0$ such that $\overline{D(z_0, r)} \subset \Omega$. By the Cauchy estimate, we have

$$|f_n'(z_0)| \leq \frac{M_n}{r},$$

where $M_n = \sup_{w \in D(z_0, r)} |f_n(w)|$. Since Ω is bounded, we conclude that M_n is bounded, and the conclusion follows.

(c) From (a) and (b) we deduce that $|f'(z_0)|^n$ is bounded, hence $|f'(z_0)| \leq 1$.

(d) By considering f^{-1}, conclude that $|f'(z_0)| = 1$.