King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH533 - Complex Variables
Comprehensive Exam – Term 231
Solution

1. (20 points)

Let $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half plane. Prove the following

(a) If $\alpha \notin H$ and $\beta \in H$, then $\frac{1}{\alpha - \beta} \in H$.

(b) For any $\xi_1, \ldots \xi_k \in H$, $\xi_1 + \ldots + \xi_k \in H$, in particular, $\xi_1 + \ldots + \xi_k \neq 0$.

(c) Let P be a polynomial with zeros $z_1, \ldots, z_k$ (possibly repeated)

$$\frac{P'(z)}{P(z)} = \frac{1}{z - z_1} + \ldots + \frac{1}{z - z_k}.$$

(d) Using the previous questions, prove the following:
" Let P be a polynomial. Suppose all the zeros of P lie in H. Then all the zeros of P' also lie in H".

Solution: (a) By assumption, we have

$$\operatorname{Im} \beta > 0 \text{ and } \operatorname{Im} \alpha \leq 0.$$

$$\frac{1}{\alpha - \beta} = \frac{\overline{\alpha} - \overline{\beta}}{|\alpha - \beta|^2}$$

Thus

$$\operatorname{Im} \frac{1}{\alpha - \beta} = \frac{\operatorname{Im} \beta - \operatorname{Im} \alpha}{|\alpha - \beta|^2} > 0$$

(b) $\operatorname{Im}(\xi_1 + \ldots, \xi_k) = \operatorname{Im}(\xi_1) \ldots + \operatorname{Im}(\xi_k)$.

(c) Write

$$P(z) = C(z - z_1) \ldots (z - z_k)$$

$$P'(z) = C(z-z_2) \ldots (z-z_k) + C(z-z_1)(z-z_3) \ldots (z-z_k) + \ldots C(z-z_1) \ldots (z-z_{k-1}).$$

Hence

$$\frac{P'(z)}{P(z)} = \frac{1}{z - z_1} + \ldots + \frac{1}{z - z_k}.$$

(d) Assume there exists w, a zero of P' and $w \notin H$. Then

$$\frac{P'(w)}{P(w)} = 0.$$

By (c), we get

$$\frac{1}{w - z_1} + \ldots + \frac{1}{w - z_k} = 0$$

We get a contradiction as for $i \in [[1..k]]$, we have

$$\frac{1}{w - z_i} \in H.$$

2. (15 points)

Let f be an analytic function on $\Omega = \{z \in \mathbb{C} : |z| < 4\}$. Suppose that $|f(z)| < 1$ on Ω. Let

$$g(z) = f(z) + z - 2.$$

(a) Prove that all the zeros of g lie in the disc $D = \{z \in \mathbb{C} : |z - 2| < 1\}$.

(b) Using Rouché theorem, prove that g has only one zero inside Ω.

(c) What is g if $\Omega = \mathbb{C}$?

Solution: (a) Let z_0 be a zero of g, then

$$z_0 - 2 = -f(z_0).$$

Thus $|z_0 - 2| = |f(z_0)| < 1$.

(b) Let $h(z) = z - 2$, trivially we have $|h(z)| = 1$ on ∂D. Moreover

$$|g(z) - h(z)| = |f(z)| < |h(z)|, \text{ on } \partial D.$$

Thus, by Rouché theorem g has only one zero inside the disc D. Since all the zeros of g are located inside D, we deduce that g has only one zero inside Ω.

(c) If $\Omega = \mathbb{C}$, by Liouville theorem, f is constant and $g(z) = z + C$, with $C \in \mathbb{C}$.

3. (10 points) Evaluate

$$\int_0^{2\pi} \frac{\cos^2\theta}{5+3\sin\theta}d\theta.$$

Solution: Let $z = e^{i\theta}$. Then

$$\cos\theta = \frac{z+\frac{1}{z}}{2}, \quad \sin\theta = \frac{z-\frac{1}{z}}{2i}, \quad d\theta = \frac{dz}{iz}.$$

Hence

$$I = \int_0^{2\pi} \frac{\cos^2\theta}{5+3\sin\theta}d\theta = \frac{1}{2}\int_{|z|=1} \frac{z^4+2z^2+1}{z^2(3z+i)(z+3i)}dz = \pi i(Res(f,i)+Res(f,-i/3)),$$

where

$$f(z) = \frac{z^4+2z^2+1}{z^2(3z+i)(z+3i)}.$$

Thus

$$I = \pi i(-\frac{10i}{9}+\frac{8i}{9}) = \frac{2\pi}{9}.$$

4. (10 points) Let Ω be a bounded domain in the complex plane. Suppose that f is continuous on $\overline{\Omega}$ and analytic on Ω. Assume $|f(z)| = 1$ for all $z \in \partial\Omega$, the boundary of Ω.
   Show that f is a constant function or f has a zero on Ω.

   Solution: Assume that f has no zeros in Ω. Then by the maximum and the minimum principle (as f has no zeros), $|f(z)|$ attains its maximum and its minimum on $\partial\Omega$. Hence
   $$|f| = 1 \quad \text{on } \overline{\Omega}.$$
   Thus f is constant on Ω.

5. (20 points) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that $\lim_{|z|\to\infty} |f(z)| = +\infty$.

(a) Prove that f has a finite number of zeros in $\mathbb{C}$.

(b) Prove that there exists a polynomial P such that $f = \frac{P}{g}$, where g is holomorphic in $\mathbb{C}$ and $g(z) \neq 0$, for all $z \in \mathbb{C}$.

(c) Prove that there exists $R > 0$ such that $|g(z)| \leq |P(z)|$, for all $|z| \geq R$ and that g is a polynomial.

(d) Deduce that there exists a constant c such that $f = cP$.

Solution: (a) By assumption, there exists $R > 0$ such that

$$|f(z)| \geq 1 \text{ if } |z| > R.$$

Hence all the zeros are inside the closed disc $\overline{D}(0, R)$. As the zeros of f are isolated, a compact set contains only a finite number. Therefore, f has a finite number of zeros.

(b) Let $a_1, \ldots, a_k$ be the zeros of f then

$$f(z) = (z - a_1)\ldots(z - a_k)h(z)$$

where h is a non-vanishing entire function. Put

$$P(z) = (z - a_1)\ldots(z - a_k),$$

$$g(z) = \frac{1}{h(z)}.$$

(c) Let R be chosen in (a). Then

$$1 < |f(z)| = \frac{|P(z)|}{|g(z)|}.$$

Hence

$$|g(z)| \leq |P(z)| \text{ for } |z| > R.$$

As P is a polynomial of degree k, then there exists $R_1 > 0$ such that

$$|P(z)| \leq C|z|^k \text{ if } |z| > R_1.$$

Thus for $|z|$ large, we have

$$|g(z)| \leq C|z|^k.$$

For all $n \geq 0$

$$g^{(n)}(0) = \frac{n!}{2\pi i} \int_{|z|=r} \frac{g(z)}{z^{n+1}}\, dz$$

$$|g^{(n)}(0)| \leq \frac{Cn!}{r^{n-k}} \to 0 \text{ as } r \to \infty, \text{ if } n \geq k+1.$$

Therefore g is a polynomial of degree $\leq k$.

(d) If g is a non constant polynomial, then g has a zero by the fundamental theorem of algebra. Since g is a non vanishing entire function, we conclude that g is constant.

6. (15 points) Let f be an analytic function on $\mathbb{C}$.

(a) Prove that for any $\alpha, \beta \in \mathbb{C}$, with $\alpha \neq \beta$, we have for $R > \max(|\alpha|, |\beta|)$

$$\frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{(z-\alpha)(z-\beta)} dz = \frac{f(\alpha) - f(\beta)}{\alpha - \beta}$$

(b) Show if f is bounded, then

$$\lim_{R\to\infty} \int_{|z|=R} \frac{f(z)}{(z-\alpha)(z-\beta)} dz = 0.$$

(c) Using ONLY (a) and (b), show that if f is analytic and bounded on $\mathbb{C}$, then f is constant. (No credit for other methods).

Solution: (a) By the Residue Theorem.

(b) Assume that

$$|f(z)| \leq M \text{ on } \mathbb{C}.$$

Then

$$\left| \int_{|z|=R} \frac{f(z)}{(z-\alpha)(z-\beta)} dz \right| \leq \frac{2\pi M R}{(R-|\alpha|)(R-|\beta|)} \to 0 \text{ as } R \to \infty.$$

(c) Combining (a) and (b), we get

$$f(\alpha) = f(\beta),$$

so f is constant.

7. (10 points) Let

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

be an analytic function and

$$F : (x,y) \mapsto (u(x,y), v(v,y)).$$

(a) Show that

$$\det J_F(x,y) = |f'(z)|^2,$$

where $J_F(x,y)$ represents the Jacobian matrix of F at (x,y).

(b) Show that if $f'(z) = 0$, then $J_F(x,y) = 0$.

Solution: (a) Using CR equations, we have

$$\det J_F(x,y) = u_x v_y - v_x u_y = u_x^2 + v_x^2 = |f'(z)|^2.$$

(b) If $f'(z) = 0$, then

$$u_x = v_x = 0.$$

Hence according to CR-equations, we obtain

$$v_y = u_x = 0, \quad u_y = -v_x = 0.$$

Therefore

$$J_F(x,y) = 0$$