King Fahd University of Petroleum and Minerals
Department of Mathematics
MATH533 - Complex Variables
Comprehensive Exam – Term 232

Name: ID:

Time Duration: 3 hrs.

Number of Questions: 7.

3 empty sheets of paper are added for your own sake.

Justify your answers thoroughly. Any theorem that you use must be quoted correctly.

1. Characterize an analytic function f on $\mathbb{C} \setminus \{0\}$ such that

- f has a pole of order 2 at 0.
- $\lim_{z\to\infty} f(z) = \infty$.

Sol Since f has a pole of order 2, $g(z) = z^2 f(z)$ has a removable singularity at 0. & $\lim_{z\to 0} g(z) \neq 0$

$\Rightarrow$ We can regard g as an entire ftn.

Since $\lim_{z\to\infty} g(z) = \infty$, g is a polynomial.

If $\deg g \leq 2$, then $\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{g(z)}{z^2} < \infty$

$\Rightarrow$ g is a polynomial of $\deg \geq 3$

$\Rightarrow$ $f(z) = \frac{g(z)}{z^2}$, where g is a poly. of $\deg \geq 3$
& $g(0) \neq 0$

Alternative answer: Since f has a pole of order 2 at 0,

$$f(z) = \sum_{n=-2}^{\infty} a_n z^n, \qquad a_{-2} \neq 0.$$

Let $g(z) = f(\frac{1}{z})$. Then $g(z) = \sum_{n=-2}^{\infty} a_n z^{-n}$

Since $\lim_{z\to 0} g(z) = \infty$, 0 is also a pole of g.

$\Rightarrow$ $a_n = 0$ if $n > N$ for some $N > 0$

$$\Rightarrow \quad f(z) = \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + \sum_{n=0}^{N} a_n z^n = \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + p(z)$$

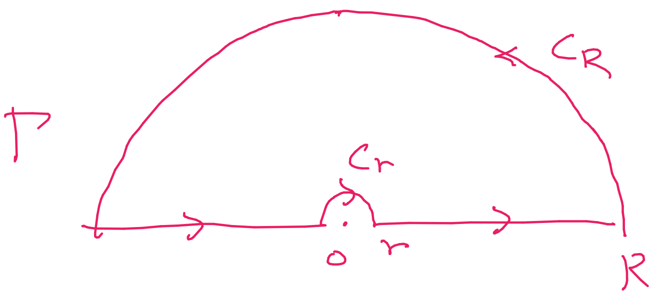
where p is a polynomial. $(a_{-2} \neq 0)$

2. Evaluate the real improper integral

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\, dx,$$

where log means the natural logarithmic function.

Let $f(z) = \dfrac{\log z}{(1+z^2)^2}$ where $\log z$ is the branch of log s.t. $\log z = \ln|z| + i \arg z$, $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$. Consider



Since f has only one sing. at $z = i$, which is a pole of order 2,

$$\int_\Gamma f(z)\,dz = 2\pi i \operatorname{Res}(f; i) = \frac{d}{dz}\bigg|_{z=i}\left(\frac{\log z}{(z+i)^2}\right)\cdot 2\pi i$$

$$= -\frac{\pi}{2} + i\frac{\pi^2}{4} \qquad \text{— (1)}$$

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{\log R + \pi}{(R^2-1)}\cdot 2\pi R \to 0 \quad \text{as } R\to\infty$$

$$\left|\int_{C_r} f(z)\,dz\right| \le \frac{\log r}{(1-r^2)^2}\cdot 2\pi r \to 0 \quad \text{as } r\to 0$$

$$\int_r^R f(z)\,dz \to \int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx =: I$$

$$\int_{-R}^{-r} f(z)\,dz = \int_{-R}^{-r} \frac{\log(-x) + i\pi}{(1+x^2)^2}\,dx$$

$$= \int_{r}^{R} \frac{\log x + i\pi}{(1+x^2)^2}\,dx$$

$$\longrightarrow I + i\pi \int_0^{\infty} \frac{dx}{(1+x^2)^2} \qquad \text{as } R\to\infty \quad r\to 0$$

$$\Rightarrow \int_P f(z)\,dz \longrightarrow 2I + i\pi \int_0^{\infty} \frac{dx}{(1+x^2)^2} \qquad \text{— (2)}$$

From (1) & (2), taking the real part

$$\boxed{I = -\frac{\pi}{4}}$$

3. Let $\Omega = \{z \in \mathbb{C} : \operatorname{Im} z > -1\}$.

(a) Find a Möbius transform which maps Ω onto the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

(b) Show that any analytic function $f : \Delta \setminus \{0\} \to \Omega$ has a removable singularity at 0.

(a) $M(z) = \dfrac{z}{z+2i}$ can be an answer.

(b) $M \circ f : \Delta \setminus \{0\} \to \Delta$ analytic

i.e. $M \circ f$ is bdd $\Rightarrow$ 0 is removable

of $M \circ f$ $\Rightarrow$ 0 is removable for f. $\square$

4. Let Δ be the unit disc in $\mathbb{C}$.

(a) Show that for any $a \in \Delta$ and $c \in \mathbb{C}$ with $|c| = 1$,

$$\varphi(z) = c\frac{z-a}{1-\bar{a}z}$$

is an automorphsm of Δ.

(b) Suppose $f \in \mathrm{Aut}(\Delta)$ such that $f(0) = 0$. Show that $f(z) = cz$ for all $z \in \Delta$, for some $c \in \mathbb{C}$ with $|c| = 1$.

(c) Using (a) and (b), show that

$$\mathrm{Aut}(\Delta) = \left\{ c\frac{z-a}{1-\bar{a}z} : a \in \Delta,\ c \in \mathbb{C},\ |c| = 1 \right\}.$$

(a) Since $\varphi(z)$ is a Möbius transform and $\varphi(a) = 0 \in \Delta$, we only need to show $|\varphi(z)| = 1$ if $|z| = 1$.

If $|z| = 1$, then $\bar{z} = 1/z$, therefore

$$|\varphi(z)| = \frac{|z-a|}{|1-\bar{a}z|} = \frac{|z|\,|1-a\bar{z}|}{|1-\bar{a}z|} = \frac{|1-a\bar{z}|}{|1-\bar{a}z|} = \frac{|\bar{w}|}{|w|} = 1$$

where $w = 1-\bar{a}z$.

(b) Applying the Schwarz lemma to both f and f^{-1}, we conclude that $|f'(0)| = 1$. Then the Schwarz lemma also implies that $f(z) = cz$ for some $c \in \mathbb{C}$, $|c| = 1$.

(c) Let $f \in \mathrm{Aut}(\Delta)$ & let $f(a) = 0$. Let $\varphi(a) = \frac{z-a}{1-\bar{a}z}$.

Then $\varphi(a) = 0 \Rightarrow \varphi^{-1}(0) = a \Rightarrow f\circ\varphi^{-1} \in \mathrm{Aut}(\Delta)$

s.t. $f\circ\varphi^{-1}(0) = 0 \underset{(b)}{\Rightarrow} f\circ\varphi^{-1}(z) = cz \Rightarrow \boxed{f(z) = c\,\varphi(z)}$ for some $c \in \mathbb{C}$, $|c| = 1$.

5. Let $f(z) = z^7 - 5z^5 + 7$. Prove that f has

(a) 5 zeros in $A_1 = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

(b) 2 zeros in $A_2 = \{z \in \mathbb{C} : 2 < |z| < 3\}$.

(i) Let $g_1(z) \equiv 7$, $h_1(z) = z^7 - 5z^5$.

Then on $|z|=1$, $|g_1(z)| = 7 > 6 \geq |h_1(z)|$

$\Rightarrow$ g_1 and $g_1 + h_1$ have same number of zeros inside $|z|=1$, which is 0

(ii) Let $g_2(z) = -5z^5$, $h_2(z) = z^7 + 7$

On $|z|=2$, $|g_2(z)| = 5 \cdot 2^5 = 160$

$|h_2(z)| \leq 2^7 + 7 = 135$

$\Rightarrow$ On $|z|=2$ $|g_2(z)| > |h_2(z)|$

$\Rightarrow$ g_2 & $f = g_2 + h_2$ have same number of zeros inside $|z| < 2$. which is 5. Since f has no zeros inside $|z| \leq 1$, this implies all these 5 zeros lie on $1 < |z| < 2$

(iii) Similarly, if $g_3(z) = z^7$, $h_3(z) = -5z^5 + 7$.

then $|g_3(z)| > |h_3(z)|$ on $|z|=3$

$\Rightarrow$ Both g_3 and $f = g_3 + h_3$ have 7 zeros inside

$|z| < 3$ & since f has five zeros inside $|z| \leq 2$, f has 2 zeros on $2 < |z| < 3$.

6. Let U be a domain in $\mathbb{C}$ and $f : U \to \mathbb{C}$ an analytc function. Let $z_0 \in U$.

(a) Prove if $f'(z_0) = 0$, then f is NOT 1-1 on $B(z_0; r)$ for any $r > 0$ such that $\overline{B(z_0; r)} \subset U$, where $B(z_0; r) = \{z \in \mathbb{C} : |z - z_0| < r\}$.

(b) Prove that if $f'(z_0) \neq 0$, then f is 1-1 on $B(z_0; r)$ for sufficiently small $r > 0$ and for $w \in f(B(z_0; r))$,

$$f^{-1}(w) = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{z f'(z)}{f(z) - w}\, dz.$$

(a) Let $\Omega = f(U)$ & let $C = \{|z - z_0| = r\}$

Let Ω_0 be the connected component of $\Omega \setminus f(C)$ containing $w_0 = f(z_0)$. Then $\forall\, w \in \Omega_0$,

The number of zeros of $f(z) - w$

$=$ index of $f(C)$ around w

$=$ 〃 w_0

$=$ the number of zeros of $f(z) - w_0 \geq 2$

$\Rightarrow$ f is not 1-1

(b) If $f'(z_0) \neq 0$, then f is 1-1 in a nbd. of z_0 by the Inverse function theorem. Suppose $r > 0$ is chosen that f is 1-1 on a nbd. of $\overline{B(z_0, r)}$.

& let $w \in f(B(z_0, r))$. Let ζ be the unique zero of $f(\cdot) - w$ in $B(z_0, r)$. i.e. $\zeta = f^{-1}(w)$. Then

$f(z) - w = (z - \zeta)\, g(z)$ where $g(z) \neq 0$ on $\overline{B(z_0, r)}$.

$$\Rightarrow \frac{1}{2\pi i}\oint_{|z-z_0|=r} \frac{z f'(z)}{f(z)-w}\, dz$$

$$= \frac{1}{2\pi i}\oint_{|z-z_0|=r} \frac{z g'(z)}{g(z)} + \frac{z}{z-\zeta}\, dz$$

$$= \quad \zeta = f^{-1}(w)$$

Since $\frac{z g'(z)}{g(z)}$ is analytic on a nbd. of $\overline{B(z_0,r)}$.

7. For a compact set K in $\mathbb{C}$, let

$$\widehat{K} = \{z \in \mathbb{C} : |f(z)| \leq \max_{w \in K} |f(w)| \text{ for all entire function } f\}.$$

A domain U in $\mathbb{C}$ is said to be *polynomially convex* if $\widehat{K} \subset U$ whenever K is a compact subset of U. Prove that the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$ is NOT polynomially convex. (Hint. Let $K = \{z \in \mathbb{C} : |z| = 3/2\}$. What is $\widehat{K}$?)

sol Any polynomial P, on $\{|z| \leq 3/2\} = D$ attains its maximum on $K = \partial D$ by the maximum modulus principle, that is,

$$|P(z)| \leq \max_{w \in K} |P(w)|$$

for any $|z| \leq 3/2$ and polynomial P.

$\Rightarrow D \subset \widehat{K}$. On the other hand, $D \not\subset A$. Therefore $\widehat{K} \not\subset A$.

$\Rightarrow A$ is not polynomially convex.