**King Fahd University of Petroleum and Minerals,**

**Department of Mathematics and Statistics**

**Comprehensive Exam: Linear Algebra (211)**

**Duration : 3 Hours**

Solve the following questions (**show full details**).

**Exercise 1 (20 points: 5-5-5-5)** Let $T$ be the linear operator on $\mathbb{R}^3$ defined by $T(x, y, z) = (x, x + y, x + z)$.

(1) Find the matrix $[T]_S$ representing $T$ in the standard basis $S$ of $\mathbb{R}^3$.

(2) Find the matrix $[T]_B$ representing $T$ in the ordered basis $B = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ of $\mathbb{R}^3$.

(3) Find the characterestic and minimal polynomials of $T$.

(4) Show that $T$ is invertible and find $T^{-1}$ (express $T^{-1}$ explicitely).

**Exercise 2 (20 points: 5-5-5-5)** Let $A$ and $B$ be $n \times n$ **complex matrices**.

(1) Show that $AB$ and $BA$ have the same nonzero eigenvalues.

(2) Assume that $AB = 0$. Show that $rank(A) + rank(B) \leq n$.

Assume that $A^s = I$ for some positive integer $s$ and let $J$ be the Jordan normal form of $A$.

$(a)$ Show that $J^s = I$.

$(b)$ Show that $A$ is diagonalizable.

**Exercise 3 (20 points:5-5-5-5)** Let $V$ be an $n$-dimensional vector space over the real field $\mathbb{R}$ with a positive definite scalar product $(|)$ and let $T$ be a linear operator on $V$ such that $T^2 = T$ and $TT^t = T^tT$, where $T^t$ is the transpose of $T$.

(1) Prove that $V = ker(T) \bigoplus Im(T)$.

(2) Prove that $ker(T) = ker(T^t)$.

(3) Prove that for every $v \in V$, $v = w + T(u)$ where $w \in ker(T)$, $||T(v)||^2 = ||T(u)||^2 = (T(v)|T^t(v))$.

(4) Prove that $T = T^t$.

**Exercise 4 (20 points)**

Find all possible rational forms and their respective Jordan forms of a matrix with characteristic polynomial $(x-1)^3(x-2)^4$ and minimal polynomial $(x-1)^2(x-2)^2$.

**Exercise 5 (20 points: 5-5-5-5)**

Let $V$ be a finite dimensional inner product space over a field $K$ and $T$ a self adjoint linear operator on $V$.

(1) Prove that each eigenvalue of $T$ is real.

(2) Prove that the eigenvectors of $T$ associated with distinct eigenvalues are orthogonal.

(3) Assume that $K = \mathbb{R}$ and let $A$ be a linear on $V$ such that $(Av|v) = 0$ for all $v \in V$. Prove that $A + A^t = 0$ and if $A$ is symmetric, then $A = 0$.

(4) Find a linear operator $A$ such that $(Av|v) = 0$ for all $v \in V$ but $A \neq 0$.

# Comprehensive Exam.
# Math 550 – Term 211
# Key Solution (by A. MIMOUNI)

**Exercise 1.** $T$ a linear operator on $\mathbb{R}^3$ defined by:

$T(x,y,z) = (x, x+y, x+z)$, $S=\{e_1,e_2,e_3\}$ standard basis.

① $[T]_S = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

② $B = \{(0,1,1), (1,0,1), (1,1,0)\} = \{u_1,u_2,u_3\}$ basis

$[T]_B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

**Work:**

$Tu_1 = (0,1,1) = u_1 = u_1 + 0u_2 + 0u_3$

$Tu_2 = (1,1,2) = u_1 + u_2 = u_1 + u_2 + 0.u_3$

$Tu_3 = (1,2,1) = u_1 + u_3 = u_1 + 0u_2 + 1u_3$

Also: $\boxed{[T]_B = P^{-1}[T]_S P}$

③ **Characteristic and minimal polynomials of $T$.**

$* \; f_T(x) = \det(xI - T) = \begin{vmatrix} x-1 & 0 & 0 \\ -1 & x-1 & 0 \\ -1 & 0 & x-1 \end{vmatrix} = (x-1)^3.$

$* \; p_T(x) = (x-1)^2.$ $\quad (A-I)^2 = 0$ but $A - I \neq 0$.

④ **$T$ invertible,** $\det[T]_S = 1 \neq 0 \Rightarrow [T]_S$ invertible and

So $T$ is invertible.

$*$ **Express $T^{-1}$ explicitly:** Let $A = [T]_S$

Clearly $p_T(A) = 0$. So $(A-I)^2 = 0 \Rightarrow A^2 - 2A + I = 0$

Then: $2A - A^2 = I$ and thus $A\underbrace{(2I - A)}_{A^{-1}} = I$.

Therefore $[T^{-1}]_S = [T]_S^{-1} = A^{-1} = 2I - A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

• **Finally,** $T^{-1}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x,y,z) \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x, -x+y, -x+z)$

**Exercise 2:** A and B complex matrices $(n\times n)$

① AB and BA have the same eigenvalues.

Let $\lambda$ be an eigenvalue of $AB$. Then $\det(AB-\lambda I)=0$

- If $\lambda=0$, then $\det(BA)=\det(AB)=0$ and so $\lambda=0$ is an eigenvalue of $BA$.
- Assume that $\lambda\neq 0$. Let $0\neq v$ such that $ABv=\lambda v$. Necessarily $Bv\neq 0$. For if not, $Bv=0$ implies that $\lambda v=ABv=A.0=0$ and so $\lambda=0$ or $v=0$, which is absurd. Hence $Bv\neq 0$. Now set $\boxed{w=Bv}$

We have. $ABv=\lambda v \Rightarrow Aw=\lambda v$
$\Rightarrow BAw=\lambda Bv=\lambda w$

So $\lambda$ is an eigenvalue of $BA$, and $w$ its associated eigenvector.

② Assume that $AB=0$. Show that $\text{rank}(A)+\text{rank}(B)\leq n$

Nullity Formula: $\text{Nullity}(M)+\text{rank}(M)=\dim V$

- Now $AB=0 \Rightarrow$ Column space $(B)\subseteq$ Null space $(A)$

So $\underbrace{\dim(\text{Column space}(B))}_{\text{rank}(B)} \leq \underbrace{\dim(\text{Null space}(A))}_{\text{Nullity}(A)}$, that is,

$\text{rank}(B)\leq \text{Nullity}(A)$.

Therefore: $\boxed{\text{rank}(A)+\text{rank}(B)\leq \text{rank}(A)+\text{Nullity}(A)=n}$

③ $A^s=I$ (for some $s\geq 1$), $J$ its Jordan Form:

ⓐ There exists an invertible matrix $Q$ such that $J=Q^{-1}AQ$. By induction on $m\geq 1$, $J^m=Q^{-1}A^mQ$.

In particular, $J^s=Q^{-1}A^sQ=Q^{-1}IQ=I$.

(b) Show that A is diagonalizable.

• Method 1: A is a complex matrix and $A^{s} = I$. then its minimal polynomial $P_A(x)$ divides $X^{s} - 1$ in $\mathbb{C}[X]$ that is, the roots of $P_A(x)$ are simple (as roots of the unity) Therefore A is diagonalizable.

• Method 2: Let $\lambda_1, \dots, \lambda_r$ be the distinct eigenvalues of A. The Jordan form is formed by r blocks

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{pmatrix}$$

$J_i$ Jordan block associated to $\lambda_i$

As $J^{s} = I$, this means $J_i^{s} = I$ for each $i$.

We reduced the proof to the case of one block J associated to $\lambda$ and $J^{s} = I$. Write $J = \lambda I + N$, N Nilpotent.

$$I = J^{s} = (\lambda I + N)^{s} = \sum_{k=0}^{s} \binom{s}{k} \lambda^{k} N^{s-k} \qquad N = (b_{ij})$$

and this shows that $b_{i\,i+1} = 0 \ \forall i$. thus $J = \lambda I$ is diagonal

**Exercise 3:** $\dim_{\mathbb{R}} V = n$, $(\ |\ )$ positive definite scalar product. T linear operator on V s.t. $T^2 = T$ and $TT^t = T^tT$.

(1) Prove that $V = \operatorname{Ker} T \oplus \operatorname{Im} T$.

Let $x \in V$. Write $x = (x - Tx) + Tx$. Clearly $Tx \in \operatorname{Im} T$. Now $T(x - Tx) = Tx - T^2x = Tx - Tx = 0$

So $x - Tx \in \operatorname{Ker} T$. Thus $V = \operatorname{Ker} T + \operatorname{Im} T$.

Now, let $x \in \operatorname{Ker} T \cap \operatorname{Im} T$. Then $Tx = 0$ and $x = Ty$ for some $y \in V$.
So $0 = Tx = T^2y = Ty = x$. Thus $\operatorname{Ker} T \cap \operatorname{Im} T = 0$
and therefore $V = \operatorname{Ker} T \oplus \operatorname{Im} T$.

② Prove that $\operatorname{Ker}(T) = \operatorname{Ker}(T^t)$.

For every $v \in V$, we have: $(T^tv \mid T^tv) = (v \mid TT^tv)$
So $\|T^tv\|^2 = (T^tv \mid T^tv) = (v \mid TT^tv) =$
$= (v \mid T^tTv)$ since $TT^t = T^tT$)
$= (Tv \mid Tv) = \|Tv\|^2$.

Therefore $\|T^tv\| = \|Tv\|$.
So $x \in \operatorname{Ker} T \iff Tx = 0 \iff \|Tx\| = 0 \iff \|T^tx\| = 0$
$\iff T^tx = 0$
$\iff x \in \operatorname{Ker} T^t$.

Hence $\boxed{\operatorname{Ker} T = \operatorname{Ker} T^t}$

③ Prove that $\forall\, v \in V$, $v = w + T(u)$, $w \in \operatorname{Ker} T$, $\|Tv\|^2 = \|Tu\|^2 = (Tv \mid T^tv)$

From ①, $V = \operatorname{Ker} T \oplus \operatorname{Im} T$. Let $v \in V$. Then $v = w + Tu$
$w \in \operatorname{Ker} T = \operatorname{Ker} T^t$. So $Tw = T^tw = 0$.
Now $v = w + Tu \Rightarrow Tv = \underbrace{Tw}_{0} + T^2u = T^2u = Tu$.
$(T = T^2)$

So $\|Tv\|^2 = (Tv \mid Tv) = (Tu \mid Tu) = \|Tu\|^2$.
Also $(Tv \mid T^tv) = (T^2v \mid v) = (Tv \mid v) = (Tu \mid v)$
$= (Tu \mid w + Tu) = (Tu \mid w) + (Tu \mid Tu) = (u \mid \underbrace{T^tw}_{0}) + \|Tu\|^2$
$= (u \mid 0) + \|Tu\|^2 = 0 + \|Tu\|^2 = \|Tu\|^2$.

④ Prove that $T = T^t$: For every $x, y \in V$

by ③ $x = w_1 + Tu_1$, $y = w_2 + Tu_2$, $\boxed{w_1, w_2 \in \ker T. \text{ So } Tw_1 = Tw_2 = 0,\ T^2 = T}$

Then: $(Tx|y) = (\underbrace{Tw_1}_{0} + T^2u_1 | y)$,

$= (Tu_1 | y) = (Tu_1 | w_2 + Tu_2)$

$= (Tu_1 | w_2) + (Tu_1 | Tu_2)$

$= (u_1 | T^t w_2) + (Tu_1 | Tu_2)$. $\boxed{T^t w_2 = 0 \text{ as } \ker T = \ker T^t}$

$= (u_1 | 0) + (Tu_1 | Tu_2)$

$= (Tu_1 | Tu_2)$ ✓

Similarly, $(T^t x | y) = (x | Ty) = (x | Tu_2)$ : $\boxed{Ty = Tu_2 \text{ as } Tw_2 = 0}$

$= (w_1 + Tu_1 | Tu_2)$

$= (w_1 | Tu_2) + (Tu_1 | Tu_2)$

$= (\underbrace{T^t w_1}_{0} | u_2) + (Tu_1 | Tu_2)$

$\boxed{w_1 \in \ker T = \ker T^t \quad Tw_1 = T^t w_1 = 0}$ $= (T^2 u_1 | Tu_2)$ ✓

Therefore, $(Tx|y) = (T^t x|y)$ and this true $\forall x, \forall y$

Thus $Tx = T^t x\ \forall x \in V$ and Hence $T = T^t$.

**Exercise 4:** T a linear operator with

$\underbrace{f_T(x) = (x-1)^3 (x-2)^4}_{\text{characteristic polynomial}}$ and $\underbrace{p_T(x) = (x-1)^2 (x-2)^2}_{\text{minimal polynomial}}$

- $\dim V = \deg f_T(x) = 7$, $p_T(x) = x^4 - 6x^3 + 13x^2 - 12x + 4$

Two cases are possible : $p = p_1$, $f = p_1 p_2 \cdots p_r$.

## Case 1 :

$$p_1 = p = (x-1)^2(x-2)^2$$

$$p_2 = (x-1)(x-2) = x^2 - 3x + 2$$

$$p_3 = (x-2)$$

### Rational Forms:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$$

$$A_3 = (2)$$

So

$$A = \left(\begin{array}{cccc|cc|c} 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 12 & 0 & 0 & 0 \\ 0 & 1 & 0 & -13 & 0 & 0 & 0 \\ 0 & 0 & 1 & 6 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{array}\right)$$

### Jordan Forms

$$J_1 = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \quad J_2 = \left(\begin{array}{cc|c|c} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array}\right)$$

- First block $2\times 2$,
- second block $1\times 1$.

- First block $2\times 2$
- Second $1\times 1$.
- Third $1\times 1$.

## Case 2 :

$$p_1 = p = (x-1)^2(x-2)^2$$

$$p_2 = (x-1)(x-2)^2$$

$$= x^3 - 5x^2 + 8x - 4.$$

### Rational Forms

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 4 \\ 1 & 0 & -8 \\ 0 & 1 & 5 \end{pmatrix}$$

So

$$A = \left(\begin{array}{cccc|ccc} 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 12 & 0 & 0 & 0 \\ 0 & 1 & 0 & -13 & 0 & 0 & 0 \\ 0 & 0 & 1 & 6 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right)$$

### Jordan Forms

$$J_1 = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \quad J_2 = \left(\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

- First block $2\times 2$
- second block $1\times 1$.

- First block: $2\times 2$
- Second block $2\times 2$

in both cases $J = \left(\begin{array}{c|c} J_1 & 0 \\ \hline 0 & J_2 \end{array}\right)$

**Exercise 5**: $\dim_K V = n < \infty$, $T$ Self-adjoint, $T = T^*$

① **Eigenvalues of $T$ are real Numbers.**

Let $\lambda$ be an eigenvalue of $T$ and $0 \neq v$ an eigenvector of $T$ associated to $\lambda$. Then $Tv = \lambda v$.

$(Tv|v) = (\lambda v|v) = \lambda(v|v) = \lambda\|v\|^2$.

Also $(Tv|v) = (v|T^*v) = (v|Tv) = (v|\lambda v)$
$= \bar{\lambda}(v|v) = \bar{\lambda}\|v\|^2$.

So $\lambda\|v\|^2 = \bar{\lambda}\|v\|^2$. As $\|v\| \neq 0$, then $\boxed{\lambda = \bar{\lambda}}$

② **Eigenvectors associated to distinct eigenvalues are $\perp$**

Let $\lambda_1 \neq \lambda_2$ distinct eigenvalues of $T$.

For every $v_1$ eigenvector of $\lambda_1$ and $v_2$ eigenvector of $\lambda_2$

$(Tv_1|v_2) = (\lambda_1 v_1|v_2) = \lambda_1(v_1|v_2)$

Also $(Tv_1|v_2) = (v_1|T^*v_2) = (v_1|Tv_2) = (v_1|\lambda_2 v_2)$
$= \bar{\lambda}_2(v_1|v_2) \overset{①}{=} \lambda_2(v_1|v_2)$

So $\lambda_1(v_1|v_2) = \lambda_2(v_1|v_2) \Rightarrow (\lambda_1 - \lambda_2)(v_1|v_2) = 0$

But $\lambda_1 \neq \lambda_2 \Longrightarrow (v_1|v_2) = 0$, as desired.

③ **Assume that $K = \mathbb{R}$. $A \in \mathcal{L}(V) \mid (Av|v) = 0\ \forall v$.**

**Prove that** $A + A^t = 0$.

For every $x, y \in V$, by hypothesis $(A(x-y)|(x-y)) = 0$

So $(Ax - Ay|x-y) = 0$. Then $(Ax|y) - (Ax|y) - (Ay|x) + (Ay|y) = 0$

$(Ax|x) = (Ay|y) = 0$, we obtain: $(Ax|y) + (Ay|x) = 0$.

So $(Ax|y) + (x|Ay) = 0$ $(K = \mathbb{R} \Rightarrow (Ay|x) = (x|Ay)$

So $(Ax|y)+(x|Ay)=0$

$\Rightarrow (Ax|y)+(A^t x|y)=0$

$\Rightarrow (Ax+A^t x|y)=0$

$\Rightarrow ((A+A^t)x|y)=0$, for evey $x, y \in V$. Therefore $(A+A^t)x=0 \ \forall x \in V$ and So $A+A^t=0$.

⊛ If $A=A^t$, then $2A=0$ and So $A=0$

④ Find a linear operator such that $(Av|v)=0$ for evey $v \in V$, but $A \neq 0$

Let $V=\mathbb{R}^2$, $T: V \longrightarrow V$.
$(x,y) \longrightarrow (y,-x)$

Clearly $[T]_S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $T \neq 0$

Now $\forall v=(x,y) \in \mathbb{R}^2=V$

$(Tv|v) = ((y,-x)|(x,y)) = yx - xy = 0$

But $T \neq 0$

Good luck.