King Fahd University of Petroleum and Minerals
College of Computing and Mathematics
Department of Mathematics

Written Comprehensive Exam (Term 212)
Linear Algebra (Duration = 3 hours)

Problem 1. Let $A := \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix}$ on $\mathbb{R}$. Determine its

(1) Characteristic polynomial f
(2) Minimal polynomial p
(3) Jordan form J
(4) Let T be a linear operator on $\mathbb{R}^3$ such that A is the matrix associated to T in the standard basis $\{e_1, e_2, e_3\}$. Show that T has a cyclic vector.

Problem 2. Let $A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$ on $\mathbb{R}$. Determine its

(1) Invariant factors $p_1, \ldots, p_r$
(2) Rational form R
(3) Let T be a linear operator on $\mathbb{R}^3$ such that A is the matrix associated to T in the standard basis $\{e_1, e_2, e_3\}$. Find an explicit cyclic decomposition of $\mathbb{R}^3$ under T; namely, find α , $\beta \in \mathbb{R}^3$ and their respective T-annihilators such that $\mathbb{R}^3 = \mathrm{Z}(\alpha, T) \oplus \mathrm{Z}(\beta, T)$.

Problem 3. **(1)** Let V be the $\mathbb{R}$-vector space of polynomials of degree ≤ 3, endowed with the inner product $(f \mid g) = \int_{-1}^{1} f(t)g(t)dt$. Let W be the subspace spanned by the monomial x^2 (i.e., $W = \mathbb{R}x^2$) and E the orthogonal projection of V on W. Let $f = a + bx + cx^2 + dx^3 \in V$. Find $E(f)$.

(2) Let V be the $\mathbb{R}$-vector space of real-valued continuous functions on the interval $[-1, 1]$, endowed with the inner product $(f \mid g) = \int_{-1}^{1} f(t)g(t)dt$. Find the orthogonal complement of the subspace of even functions.

Problem 4. Let V be a finite-dimensional vector space over $\mathbb{R}$ and let L_1 and L_2 be two *nonzero* linear functionals on V. Consider the bilinear form on V given by

$$f(\alpha,\beta) = L_1\alpha\, L_2\beta$$

(1) Show that $\operatorname{rank}(f) = 1$.

Next, let $V = \mathbb{R}^3$ and let

$$L_1 : \begin{array}{ccc} V & \longrightarrow & \mathbb{R} \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto & x-y \end{array} \quad ; \quad L_2 : \begin{array}{ccc} V & \longrightarrow & \mathbb{R} \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto & x-z \end{array}$$

(2) Find the matrix of f in the standard ordered basis $S := \{e_1, e_2, e_3\}$.

(3) Let $B := \{\alpha_1, \alpha_2, \alpha_3\}$ be an ordered basis for V such that the transition matrix from B to S is

$$P = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

Find the matrix of f in B

(4) Is f *non-degenerate* ? (Justify)

Problem 5. Let V be a finite-dimensional vector space over a field F and T a linear operator on V. Let $p = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$ be the minimal polynomial of T, where $r_i \geq 1$ for each i and $p_1, p_2, \dots, p_k$ are distinct monic irreducible polynomials in $F[x]$. For each $i = 1, \dots, k$, set $W_i := \operatorname{Nullspace}\left(p_i^{r_i}(T)\right)$.

(1) Announce the Primary Decomposition Theorem.

(2) For each i, prove there exists $\alpha_i \in W_i$ such that the T-annihilator of α_i is equal to $p_i^{r_i}$.

(3) Use (2) to prove there exists $\alpha \in V$ such that the T-annihilator of α is equal to p.

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KEY

Problem 1. [10]

Let $A := \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix}$ on $\mathbb{R}$. Determine its

(1) Characteristic polynomial f

(2) Minimal polynomial p

(3) Jordan form J

(4) Let T be a linear operator on $\mathbb{R}^3$ such that A is the matrix associated to T in the standard basis $\{e_1, e_2, e_3\}$. Show that T has a cyclic vector.

(1) •• $f = \det(xI - A) = (x-1)(x-2)(x+2)$ (A is diagonalizable)

(2) •• $p = f$ (since f and p share the same roots)

(3) •• $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ (since A is diagonalizable)

(There are six versions for J depending on the order of the characteristic values 1, 2, -2)

(4) . $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; $Te_3 = 2e_1 + e_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$; $T^2e_3 = 4e_1 + e_2 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$

. • e_3 , Te_3 , T^2e_3 Linearly Independent

. • $\mathbb{R}^3 = Z(e_3, T)$;

. •• e_3 is a cyclic vector.

Problem 2. [10]

Let $A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$ on $\mathbb{R}$. Determine its

(1) Invariant factors $p_1, \dots, p_r$

(2) Rational form R

(3) Let T be a linear operator on $\mathbb{R}^3$ such that A is the matrix associated to T in the standard basis $\{e_1, e_2, e_3\}$. Find an explicit cyclic decomposition of $\mathbb{R}^3$ under T; namely, find α , $\beta \in \mathbb{R}^3$ and their respective T-annihilators such that $\mathbb{R}^3 = Z(\alpha, T) \oplus Z(\beta, T)$.

(1) $xI - A \sim \begin{pmatrix} (x-1)(x-2) & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

•• •• $p_1 = p = (x-1)(x-2) = x^2 - 3x + 2$; $p_2 = x - 1$

(2) •• $R = \left(\begin{array}{cc|c} 0 & -2 & 0 \\ 1 & 3 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$

(3) We have a cyclic decomposition **:**

$$\left\langle \begin{array}{l} \bullet\ \bullet\quad \mathbb{R}^3 = Z(e_1, T) \oplus Z(e_2 + e_3, T) \\ \bullet\ \bullet\quad p_{e_1} = p_1 \quad ; \quad p_{e_2+e_3} = p_2 \end{array} \right.$$

Indeed, first, recall that for any vector α, its T-annihilator $p_\alpha \mid p = (x-1)(x-2)$.

. $Te_1 = e_1 + e_3 \Longrightarrow p_{e_1} = p_1 \Longrightarrow \{e_1, Te_1\}$ basis for $Z(e_1, T)$

. $\beta := e_2 + e_3 : T\beta = \beta \Longrightarrow p_\beta = p_2 \Longrightarrow \{\beta\}$ basis for $Z(\beta, T)$

. $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; Te_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; \beta = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ Linearly Independent

Another cyclic decomposition **:** $\left\langle \begin{array}{l} \mathbb{R}^3 = Z(e_2, T) \oplus Z(e_1 - e_3, T) \\ p_{e_2} = p_1 \quad ; \quad p_{e_1-e_3} = p_2 \end{array} \right.$

Problem 3. [10]

(1) Let V be the $\mathbb{R}$-vector space of polynomials of degree ≤ 3, endowed with the inner product $(f \mid g) = \int_{-1}^{1} f(t)g(t)dt$. Let W be the subspace spanned by the monomial x^2 (i.e., $W = \mathbb{R}x^2$) and E the orthogonal projection of V on W. Let $f = a + bx + cx^2 + dx^3 \in V$. Find $E(f)$.

(2) Let V be the $\mathbb{R}$-vector space of real-valued continuous functions on the interval $[-1,1]$, endowed with the inner product $(f \mid g) = \int_{-1}^{1} f(t)g(t)dt$. Find the orthogonal complement of the subspace of even functions.

(1) Recall that if $\{\alpha_1, \ldots, \alpha_k\}$ is an orthogonal basis for W, then for any $\alpha \in V$, the orthogonal projection (best approximation) of α on W is given by

. • $$E\alpha = \sum_{i=1}^{k} \frac{(\alpha \mid \alpha_i)}{\|\alpha_i\|^2} \alpha_i$$

In our case, $W = \mathbb{R}x^2$ so that, for any $f = a + bx + cx^2 + dx^3 \in V$, we have:

. • $$Ef = \frac{(f \mid x^2)}{\|x^2\|^2} x^2$$

. • $$(f \mid x^2) = \int_{-1}^{1} f(t)t^2 dt = \int_{-1}^{1} \left(at^2 + bt^3 + ct^4 + dt^5\right)dt = \left[\frac{1}{3}at^3 + \frac{1}{4}bt^4 + \frac{1}{5}ct^5 + \frac{1}{6}dt^6\right]_{-1}^{1} = \left(\frac{2}{3}a + \frac{2}{5}c\right)$$

. • $$\|x^2\|^2 = (x^2 \mid x^2) = \frac{2}{5}$$

. • $$Ef = \left(\frac{5}{3}a + c\right)x^2$$

(2) Let W_e and W_o denote, respectively, the subspaces of V of even and odd functions.

. •• $W_o \subseteq W_e^{\perp}$ **:** Let $f \in W_o$. Then, for any $g \in W_e$, we have

$$
\begin{aligned}
(f \mid g) &= \int_{-1}^{1} f(t)g(t)dt \\
&= -\int_{-1}^{1} f(-t)g(t)dt \quad \big(f(t) = -f(-t)\big) \\
&= -\int_{1}^{-1} f(u)g(-u)(-du) \qquad (u = -t) \\
&= -\int_{-1}^{1} f(u)g(u)du \quad \big(g(-u) = g(u)\big) \\
&= -(f \mid g)
\end{aligned}
$$

Hence $(f \mid g) = 0$. That is, $f \in W_e^{\perp}$.

. •• $V = W_e \oplus W_o$ **:**

$W_e \cap W_o = 0$: Obvious

$W = W_e + W_o : \forall\, f \in V\,,\; f(x) = \overbrace{\frac{1}{2}\big(f(x) + f(-x)\big)}^{\in W_e} + \overbrace{\frac{1}{2}\big(f(x) - f(-x)\big)}^{\in W_o}$

. • $W_o = W_e^{\perp}$ **:**

$$
\left.
\begin{aligned}
&V = W_e \oplus W_o \\
&W_o \subseteq W_e^{\perp} \\
&W_e^{\perp} \cap W_e = 0
\end{aligned}
\right\} \implies W_e^{\perp} = W_o
$$

Problem 4. [10]

Let V be a finite-dimensional vector space over $\mathbb{R}$ and let L_1 and L_2 be two *nonzero* linear functionals on V. Consider the bilinear form on V given by

$$f(\alpha,\beta) = L_1\alpha\, L_2\beta$$

(1) Show that $\text{rank}(f) = 1$.

Next, let $V = \mathbb{R}^3$ and let

$$L_1 : \begin{array}{ccc} V & \longrightarrow & \mathbb{R} \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto & x-y \end{array} \quad ; \quad L_2 : \begin{array}{ccc} V & \longrightarrow & \mathbb{R} \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto & x-z \end{array}$$

(2) Find the matrix of f in the standard ordered basis $S := \{e_1, e_2, e_3\}$.

(3) Let $B := \{\alpha_1, \alpha_2, \alpha_3\}$ be an ordered basis for V such that the transition matrix from B to S is

$$P = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

Find the matrix of f in B

(4) Is f *non-degenerate* ? (Justify)

(1) . Let $L_f : \begin{array}{ccc} V & \longrightarrow & V^\star \\ \alpha & \mapsto & L_f\alpha : \begin{array}{ccc} V & \longrightarrow & F \\ \beta & \mapsto & f(\alpha,\beta) \end{array} \end{array}$

$$\begin{aligned} L_f\alpha = 0 \quad &\Longleftrightarrow \quad f(\alpha,\beta) = 0\,,\ \forall\,\beta \\ &\Longleftrightarrow \quad L_1\alpha\, L_2\beta = 0\,,\ \forall\,\beta \\ &\Longleftrightarrow \quad L_1\alpha = 0 \quad (\text{since } L_2 \neq 0) \end{aligned}$$

●● ... so that $\text{nullity}(L_f) = \text{nullity}(L_1)$.

. Then

$$
\begin{aligned}
\bullet\ \operatorname{rank}(f) &= \operatorname{rank}(L_f) \\
\bullet\ \ &= \dim(V) - \operatorname{nullity}(L_f) \\
&= \dim(V) - \operatorname{nullity}(L_1) \\
&= \operatorname{rank}(L_1) \\
\bullet\ \ &= 1 \quad (\text{since } L_1 \text{ is a linear functional})
\end{aligned}
$$

(2)

$$
\begin{aligned}
\bullet\ \left[f\right]_S &= \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & f(e_i, e_j) & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \\
\bullet\ \ &= \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
\qquad \text{since} \quad \begin{cases} L_1 e_1 = 1 & | \quad L_2 e_1 = 1 \\ L_1 e_2 = -1 & | \quad L_2 e_2 = 0 \\ L_1 e_3 = 0 & | \quad L_2 e_3 = -1 \end{cases}
$$

(3)

$$
\begin{aligned}
\bullet\ \left[f\right]_B &= P^t \left[f\right]_S P \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \\
\bullet\ \ &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{pmatrix}
\end{aligned}
$$

(4) Answer : NO.

• f is degenerate (or singular) since its matrix is singular $\big($equivalently, since L_f is singular. e.g., $L_f e_3 = 0$ though $e_3 \neq 0\big)$.

Problem 5. [10]

Let V be a finite-dimensional vector space over a field F and T a linear operator on V. Let $p = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$ be the minimal polynomial of T, where $r_i \geq 1$ for each i and $p_1, p_2, \dots, p_k$ are distinct monic irreducible polynomials in $F[x]$. For each $i = 1, \dots, k$, set $W_i := \text{Nullspace}\big(p_i^{r_i}(T)\big)$.

(1) Announce the Primary Decomposition Theorem.

(2) For each i, prove there exists $\alpha_i \in W_i$ such that the T-annihilator of α_i is equal to $p_i^{r_i}$.

(3) Use (2) to prove there exists $\alpha \in V$ such that the T-annihilator of α is equal to p.

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(1) Under the above notation, the Primary Decomposition Theorem asserts that

(a) •• $V = \bigoplus_{i=1}^{k} W_i$

(b) • W_i is invariant under T, $\forall\, i = 1, \dots, k$

(c) • $\text{Min. Poly.}(T_{W_i}) = p_i^{r_i}$, $\forall\, i = 1, \dots, k$

Throughout, we shall denote by p_α the T-annihilator of α.

(2) • • •

$$\text{Min. Poly.}(T_{W_i}) \overset{\text{by (c)}}{=} p_i^{r_i} \implies \forall\, \alpha \in W_i,\ p_i^{r_i}(T)\alpha = 0$$

$$\implies \exists\, \alpha_i \in W_i \ \ s.t.\ \ p_i^{r_i-1}(T)\alpha_i \neq 0 \quad \text{(Minimality)}$$

$$\implies p_{\alpha_i} \mid p_i^{r_i} \quad \text{but} \quad p_{\alpha_i} \nmid p_i^{r_i-1}$$

$$\implies \boxed{p_{\alpha_i} = p_i^{r_i}} \quad \text{(since } p_i \text{ is monic irreducible)}$$

(3) • • •

Let $\alpha := \sum_{i=1}^{k} \alpha_i$, the α_i's from **(2)** .

$$p_\alpha(T)\alpha = 0 \implies \sum_{i=1}^{k} \underbrace{p_\alpha(T)\alpha_i}_{\in W_i \text{ by (b)}} = 0$$

$$\implies p_\alpha(T)\alpha_i = 0 \text{ , for each } i \text{ , by (a)}$$

$$\overset{\text{by (2)}}{\implies} p_i^{r_i} = p_{\alpha_i} \mid p_\alpha \text{ , for each } i$$

$$\implies p = p_1^{r_1} \cdots p_k^{r_k} \mid p_\alpha$$

$$\implies \boxed{p = p_\alpha} \quad \text{since always } p_\alpha \mid p$$
