King Fahd University of Petroleum and Minerals
College of Computing and Mathematics
Department of Mathematics

Written Comprehensive Exam (Term 221)

Linear Algebra (Duration = 2 hours)

Name: ———————————————- ID number: ———————————

Problem 1. Let T_1, T_2, and T_3 be linear operators on $\mathbb{R}^3$ represented in the standard basis $\{e_1, e_2, e_3\}$, respectively, by the following matrices

$$A_1 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix} \quad ; \quad A_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad ; \quad A_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

For each T_i , find

(a) the characteristic polynomial,
(b) the characteristic values,
(c) the minimal polynomial.
(d) Determine if it is diagonalizable. If affirmative, find a basis B such that $[T_i]_B$ is diagonal.
(e) Find a cyclic decomposition of $\mathbb{R}^3$ under T_i

Problem 2. Let M be a matrix with characteristic polynomial $f = x^3(x-1)^4$ and minimal polynomial $p = x^2(x-1)^2$. Find for M all possible rational forms and their respective Jordan forms

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Problem 3. Let $V = F^{n\times n}$ be the vector space of $n \times n$ matrices over a field F and let $B \in V$.

(a) Show that the function f_B defined on V by $f_B(A) = \operatorname{trace}(B^t A)$ is a linear functional.
(b) Show that every linear functional f on V is of the form $f = f_B$ for some B in V.
(c) Show that the mapping $\phi : V \longrightarrow V^\star$, $A \mapsto f_A$ is an isomorphism.

Problem 4. Let V be a finite-dimensional vector space over a field F of characteristic 0 and let f be a *symmetric* bilinear form on V. For each subspace W of V, let $W^\perp$ be the subspace of all vectors α in V such that $f(\alpha, \beta) = 0$ for every β in W. Show that

(a) f is non-degenerate if and only if $V^\perp = 0$
(b) $\operatorname{rank}(f) = \dim(V) - \dim(V^\perp)$.
(c) $\dim(W^\perp) \geq \dim(V) - \dim(W)$.
(d) $V = W \oplus W^\perp$ if and only if the restriction of f to W is non-degenerate.

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KEY

Problem 1. [30]

Let T_1, T_2, and T_3 be linear operators on $\mathbb{R}^3$ represented in the standard basis $\{e_1, e_2, e_3\}$, respectively, by the following matrices

$$A_1 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix} \quad ; \quad A_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad ; \quad A_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

For each T_i, find

(a) the characteristic polynomial,
(b) the characteristic values,
(c) the minimal polynomial.
(d) Determine if it is diagonalizable. If affirmative, find a basis B such that $[T_i]_B$ is diagonal.
(e) Find a cyclic decomposition of $\mathbb{R}^3$ under T_i

- $A_1 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$ [10 points]

- (a) Charac. Poly.(A_1) = $\det(xI - A_1) = (x-1)(x-2)(x+1) = x^3 - 2x^2 - x + 2$ (A_1 is triangular)

- (b) Characteristic values: $c_1 = 1$, $c_2 = 2$, and $c_3 = -1$

- (c) Min. Poly.(A_1) = Charac. Poly.(A_1) = $(x-1)(x-2)(x+1)$ since they share the same roots

- (d) A_1 is diagonalizable since its minimal polynomial is a product of distinct linear factors

Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

- $c_1 = 1 \quad ; \quad A_1 X = X \quad ; \quad X = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$. Let $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

- $c_2 = 2$; $A_1 X = 2X$; $X = \begin{pmatrix} x \\ x \\ 0 \end{pmatrix}$. Let $X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

- $c_3 = -1$; $A_1 X = -X$; $X = \begin{pmatrix} x \\ -2x \\ 2x \end{pmatrix}$. Let $X_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

Let $B := \{X_1, X_2, X_3\}$. Then, $[T_1]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(e) • Since Min. Poly.(A_1) = Charac. Poly.(A_1), T_1 has a cyclic vector.

- $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $Te_3 = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$, $T^2 e_3 = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$ Linearly Independent (that is, e_3 is a cyclic vector)

- $\mathbb{R}^3 = Z(e_3, T)$;

- $A_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ [8 points]

- (a) Charac. Poly.(A_2) = $\det(xI - A_2) = x^3$

- (b) Characteristic value: $c = 0$

•• (c) Since $A_2^2 \neq 0$, Min. Poly.(A_2) = x^3 by Cayley-Hamilton Theorem

- (d) A_2 is NOT diagonalizable as Min. Poly.(A_2) is NOT a product of distinct linear factors

(e) • Since Min. Poly.(A_2) = Charac. Poly.(A_2), T_2 has a cyclic vector.

- e_2, $Te_2 = e_3$, $T^2 e_2 = Te_3 = e_1$ Linearly Independent (that is, e_2 is a cyclic vector)

- $\mathbb{R}^3 = Z(e_2, T)$

- $A_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$ [12 points]

- (a) Charac. Poly.(A_3) = $\det(xI - A_3) = (x-1)^2(x-2) = x^3 - 4x^2 + 5x - 2$

- (b) Characteristic values: $c_1 = 1$, $c_2 = 2$

•• (c) Since $(A_3 - I)(A_3 - 2I) = 0$, Min.Poly.$(A_3) = (x-1)(x-2) = x^2 - 3x + 2$

• (d) A_3 is diagonalizable since the minimal polynomial is a product of distinct linear factors

Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

• $c_1 = 1 \quad ; \quad A_3 X = X \quad ; \quad X = \begin{pmatrix} x \\ y \\ y-x \end{pmatrix}$. Let $X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

• $c_2 = 2 \quad ; \quad A_3 X = 2X \quad ; \quad X = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$. Let $X_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Let $B := \{X_1, X_2, X_3\}$. Then, $[T_1]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(e) • The invariant factors are : $p_1 = \text{Min.Poly.}(T_3) = x^2 - 3x + 2 \quad ; \quad p_2 = x - 1$

... and recall that for any vector α, its T-annihilator $p_\alpha \mid p_1$.

• $\{e_1, Te_1\}$ basis for $Z(e_1, T)$ since $Te_1 = e_1 + e_3$ and so $p_{e_1} = p_1$

• For $\beta := e_2 + e_3$, $\{\beta\}$ basis for $Z(\beta, T)$ since $T\beta = \beta$ and so $p_\beta = p_2$

• $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $Te_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$; $\beta = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ Linearly Independent

• $\mathbb{R}^3 = Z(e_1, T) \oplus Z(e_2 + e_3, T) \quad \big($or $\quad \mathbb{R}^3 = Z(e_2, T) \oplus Z(e_1 - e_3, T)\big)$

Problem 2. [30]

Let M be a matrix with characteristic polynomial $f = x^3(x-1)^4$ and minimal polynomial $p = x^2(x-1)^2$. Find for M all possible rational forms and their respective Jordan forms

Case 1 : [3 points]

$$\begin{cases} \bullet\ f = p_1 p_2 \text{ with } p_2 | p_1 \\ \bullet\ p_1 = p = x^2(x-1)^2 \ = \ x^4 - 2x^3 + x^2 \\ \bullet\ p_2 = x(x-1)^2 \ = \ x^3 - 2x^2 + x \end{cases}$$

Rational Form : [4 points]

$$\bullet\bullet\bullet\bullet\ \mathcal{R}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Jordan Form : [8 points]

Characteristic Values $\begin{cases} c_1 = 0 & \text{with } d_1 = 3, r_1 = 2 \\ c_2 = 1 & \text{with } d_2 = 4, r_2 = 2 \end{cases}$

$\mathcal{J}_1 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, where $\begin{cases} A_1 & 3\times 3 \text{ matrix} \\ A_2 & 4\times 4 \text{ matrix} \end{cases}$

$\bullet\bullet\bullet$ For $c_1 = 0$: $J_1^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ since $r_1 = 2$; $J_2^{(1)} = (0)$; $A_1 = \begin{pmatrix} J_1^{(1)} & 0 \\ 0 & J_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\bullet$ For $c_2 = 1$: $J_1^{(2)} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ since $r_2 = 2$

- $J_2^{(2)} = J_1^{(2)}$ since nullity$(\mathcal{R}_1 - I) = 2$ as the characteristic space of 1 is generated by

$$\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

- $A_2 = \begin{pmatrix} J_1^{(2)} & 0 \\ 0 & J_2^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

$$\bullet\bullet\ \mathcal{J}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 1 \end{pmatrix}$$

Case 2 : [3 points]

- $f = p_1 p_2 p_3$ with $p_3 | p_2 | p_1$

$p_1 = x^2(x-1)^2 = x^4 - 2x^3 + x^2$

- $p_2 = x(x-1) = x^2 - x$
- $p_3 = x - 1$

Rational Form : [4 points]

$$\bullet\bullet\bullet\bullet\ \mathcal{R}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Jordan Form : [8 points]

Similarly, as above, Characteristic Values $\begin{cases} c_1 = 0 & \text{with } d_1 = 3, r_1 = 2 \\ c_2 = 1 & \text{with } d_2 = 4, r_2 = 2 \end{cases}$

$\mathcal{J}_2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, where $\begin{cases} A_1 & 3\times 3 \text{ matrix} \\ A_2 & 4\times 4 \text{ matrix} \end{cases}$

- ••• For $c_1 = 0$: $J_1^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ since $r_1 = 2$; $J_2^{(1)} = (0)$; $A_1 = \begin{pmatrix} J_1^{(1)} & 0 \\ 0 & J_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- For $c_2 = 1$: $J_1^{(2)} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ since $r_2 = 2$

- $J_2^{(2)} = J_3^{(2)} = (1)$ since nullity$(\mathcal{R}_2 - I) = 3$ as the characteristic space of 1 is generated by

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} ; \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- $A_2 = \begin{pmatrix} J_1^{(2)} & 0 & 0 \\ 0 & J_2^{(2)} & 0 \\ 0 & 0 & J_3^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

$$\bullet\bullet\ \mathcal{J}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem 3. [20]

Let $V = F^{n\times n}$ be the vector space of $n\times n$ matrices over a field F and let $B \in V$.

(a) Show that the function f_B defined on V by $f_B(A) = \text{trace}(B^t A)$ is a linear functional.
(b) Show that every linear functional f on V is of the form $f = f_B$ for some B in V.
(c) Show that the mapping $\phi : V \longrightarrow V^\star$, $A \mapsto f_A$ is an isomorphism.

(a)

$$\begin{aligned} f_B(A + cA') &= trace(B^t(A + cA')) \\ &= trace(B^t A + cB^t A') \\ &= trace(B^t A) + c\, trace(B^t A'), \quad \text{since } trace \text{ is a linear functional} \\ &= f_B(A) + c\, f_B(A') \end{aligned}$$

••

(b) Let $E_{rs} = \begin{pmatrix} . & . & . \\ . & e_{ij} & . \\ . & . & . \end{pmatrix}$ denote the $n\times n$ matrix with $e_{rs} = 1$ and $e_{ij} = 0\ \forall i \neq r\ \forall j \neq s$.

•• $\left(E_{rs}\right)_{r,s}$ is a basis for V.

Let $A = \begin{pmatrix} . & . & . \\ . & a_{ij} & . \\ . & . & . \end{pmatrix} \in V$.

•• $A = \sum_{i,j} a_{ij} E_{ij}$

Let $f \in V^\star$.

•• $f(A) = \sum_{i,j} a_{ij} f(E_{ij})$ (Property (3) of the dual basis - Section 3.5)

•• Take $B := \begin{pmatrix} . & . & . \\ . & f(E_{ji}) & . \\ . & . & . \end{pmatrix}$ so that $B^t = \begin{pmatrix} . & . & . \\ . & f(E_{ij}) & . \\ . & . & . \end{pmatrix}$

•• $f_B(A) = trace(B^t A) = \sum_{1\le j\le n} \sum_{1\le i\le n} f(E_{ij}) a_{ij} = \sum_{i,j} a_{ij} f(E_{ij}) = f(A)$

(c) •• ϕ is well-defined by (a)

•• ϕ is onto by (b)

•• ϕ is a linear transformation since for any $X \in V$:
$f_{A_1+cA_2}(X) = trace\left((A_1 + cA_2)^t X\right) = trace(A_1^t X) + c\; trace(A_2^t X) = f_{A_1}(X) + c\; f_{A_2}(X)$

•• ϕ is an isomorphism since $\dim(V) = \dim(V^\star)$ (*i.e.*, $= n^2$)

Problem 4. [20]

Let V be a finite-dimensional vector space over a field F of characteristic 0 and let f be a *symmetric* bilinear form on V. For each subspace W of V, let $W^{\perp}$ be the subspace of all vectors α in V such that $f(\alpha,\beta) = 0$ for every β in W.

Show that

(a) [2] f is non-degenerate if and only if $V^{\perp} = 0$
(b) [6] $\text{rank}(f) = \dim(V) - \dim(V^{\perp})$.
(c) [6] $\dim(W^{\perp}) \geq \dim(V) - \dim(W)$.
(d) [6] $V = W \oplus W^{\perp}$ if and only if the restriction of f to W is non-degenerate.

(a)

$$f \text{ is non-degenerate} \iff \forall\, 0 \neq \alpha \in V,\ \exists\, \beta \in V \text{ s.t. } f(\alpha,\beta) \neq 0$$
$$\iff V^{\perp} = 0$$

(b) Let $n := \dim(V)$ and $r := \text{rank}(f)$. We prove: $\dim(V^{\perp}) = n - r$.

Method 1: Since f is symmetric, there is an ordered basis $B = \{\alpha_1, \ldots, \alpha_n\}$ s.t. $[f]_B$ is diagonal

Mutatis mutandis, we may assume $\begin{cases} f(\alpha_i,\alpha_i) \neq 0\ \forall\ 1 \leq i \leq r \\ f(\alpha_i,\alpha_i) = 0\ \forall\ r+1 \leq i \leq n \\ f(\alpha_i,\alpha_j) = 0\ \forall\ i \neq j \end{cases}$

Claim: $V^{\perp} = \langle \alpha_{r+1}, \ldots, \alpha_n \rangle$ (and we're done)

$(\supseteq)$ Let $i \in \{r+1, \ldots, n\}$ and let $\beta = \sum_{j=1}^{n} c_j \alpha_j \in V$.

$f(\alpha_i,\beta) = \sum_{j=1}^{n} c_j\, f(\alpha_i,\alpha_j) = 0$ so that $\alpha_i \in V^{\perp}$

$(\subseteq)$ Let $\alpha = \sum_{j=1}^{n} c_j \alpha_j \in V^{\perp}$ and let $i \in \{1, \ldots, r\}$.

$0 = f(\alpha,\alpha_i) = \sum_{j=1}^{n} c_j f(\alpha_j,\alpha_i) = c_i f(\alpha_i,\alpha_i)$

Hence $c_i = 0$ (as $f(\alpha_i,\alpha_i) \neq 0$) and so $\alpha \in \langle \alpha_{r+1}, \ldots, \alpha_n \rangle$

Method 2:

Consider the linear transformation
$$\begin{array}{rccl} \mathrm{L}_f: & V & \longrightarrow & V^\star \\ & \alpha & \mapsto & \mathrm{L}_f\alpha : \begin{array}{ccl} V & \longrightarrow & F \\ \beta & \mapsto & \mathrm{L}_f\alpha(\beta) = f(\alpha,\beta) \end{array} \end{array}$$

$$\begin{aligned} \text{We have } \dim(V) &= \operatorname{rank}(\mathrm{L}_f) + \operatorname{nullity}(\mathrm{L}_f) \\ &= \operatorname{rank}(f) + \operatorname{nullity}(\mathrm{L}_f) \end{aligned}$$

$$\begin{aligned} \text{But Nullspace}(\mathrm{L}_f) &= \left\{\alpha \in V \mid \mathrm{L}_f(\alpha) = 0\right\} \\ &= \left\{\alpha \in V \mid f(\alpha,\beta) = 0 \ \forall\, \beta \in V\right\} \\ &= V^\perp \end{aligned}$$

Therefore, $\dim(V^\perp) = \operatorname{nullity}(\mathrm{L}_f) = \dim(V) - \operatorname{rank}(f)$.

(c) Let $n := \dim(V)$, $m := \dim(W)$, and $\left\{\beta_1,\ldots,\beta_m\right\}$ be a basis for W.

Consider the linear transformation
$$\begin{array}{rccl} \varphi: & V & \longrightarrow & F^m \\ & \alpha & \mapsto & \big(f(\alpha,\beta_1),\ldots,f(\alpha,\beta_m)\big) \end{array}$$

$$\begin{aligned} \varphi(\alpha) = 0 \quad &\Longleftrightarrow \quad f(\alpha,\beta_i) = 0\,,\ \forall\, i = 1,\ldots,m \\ &\Longleftrightarrow \quad f(\alpha,\beta) = 0\,,\ \forall\, \beta \in W \\ &\Longleftrightarrow \quad \alpha \in W^\perp \end{aligned}$$

Hence Nullspace$(\varphi) = W^\perp$ so that
$$\dim(W^\perp) = n - \operatorname{rank}(\varphi) \geq n - m$$

(d) First, we have
$$\begin{aligned} f\big|_W \text{ is non-degenerate} \quad &\Longleftrightarrow \quad \forall\, 0 \neq \alpha \in W,\ \exists\, \beta \in W \ s.t.\ f(\alpha,\beta) \neq 0 \\ &\Longleftrightarrow \quad W \cap W^\perp = 0 \end{aligned}$$

Claim: $W \cap W^\perp = 0 \Longrightarrow V = W \oplus W^\perp$ (and we're done)

$$\text{Let } \left\langle \begin{array}{l} m = \dim(W) \text{ and } \left\{\beta_1,\ldots,\beta_m\right\} \text{ be a basis for } W \\ s = \dim(W^\perp) \text{ and } \left\{\gamma_1,\ldots,\gamma_s\right\} \text{ be a basis for } W^\perp \end{array} \right.$$

$$
\begin{aligned}
W \cap W^{\perp} = 0 \quad &\Longrightarrow \quad \{\beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_s\} \text{ Lin. Indep.} \\
&\Longrightarrow \quad m + s \leq n \\
&\Longrightarrow \quad m + s = n \text{ by (c)} \\
&\Longrightarrow \quad V = W \oplus W^{\perp}
\end{aligned}
$$