King Fahd University of Petroleum and Minerals
College of Computing and Mathematics
Department of Mathematics

Written Comprehensive Exam (Term 232)

Linear Algebra (Duration = 3 hours | Max. Score = 100)

KEY

Problem 1. [15] [**Section 3.4 - Ex 6 modified**]

Let T be the linear operator on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (-x_2, x_1)$

(1) [5] Consider the ordered basis $B = \{\alpha_1, \alpha_2\}$, with $\alpha_1 = (1,2)$ and $\alpha_2 = (1,-1)$. Find $[T]_B$ the matrix of T in B.

(2) [5] Prove that if B is **any** ordered basis for $\mathbb{R}^2$ and $[T]_B = (a_{ij})$, then $a_{12}a_{21} \neq 0$.

(3) [5] Let W be a nonzero proper subspace of $\mathbb{R}^2$. Prove that W is NOT T-invariant.

SOLUTION

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(1) • The matrix of T in the standard basis S is given by $[T]_S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

•••• $P = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$; $[T]_B = P^{-1}[T]_S P = \frac{1}{-3}\begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -1/3 & 2/3 \\ -5/3 & 1/3 \end{pmatrix}$.

Or $\left(P \,\middle|\, [T]_S P\right) = \left(\begin{array}{cc|cc} 1 & 1 & -2 & 1 \\ 2 & -1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & -2 & 1 \\ 0 & -3 & 5 & -1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & -5/3 & 1/3 \end{array}\right)$; $[T]_B = \begin{pmatrix} -1/3 & 2/3 \\ -5/3 & 1/3 \end{pmatrix}$.

(2) • Let $P = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, the transition matrix from B to S ; $ad - bc \neq 0$.

•• $[T]_B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = P^{-1}[T]_S P = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} -(ac+bd) & -(c^2+d^2) \\ a^2+b^2 & ac+bd \end{pmatrix}$.

$a_{12}a_{21} = 0 \implies c^2 + d^2 = 0 \;\; or \;\; a^2 + b^2 = 0$

•• $\implies c = d = 0 \;\; or \;\; a = b = 0$

$\implies ad - bc = 0$, absurd.

(3) • $\text{Charac.Poly.}(T) = \det\left([T]_S - xI\right) = x^2 + 1$, and so T has no real characteristic values.

• W is a nonzero proper subspace of $\mathbb{R}^2$, then $\dim(W) = 1$.

• There is $0 \neq \alpha \in \mathbb{R}^2$ such that $W = \langle \alpha \rangle$.

•• If W is T-invariant, then $T\alpha = c\alpha$ for some $c \in \mathbb{R}$, absurd.

Problem 2. [15] [Section 3.5 - Ex 3, 17]

Let W be the space of $n\times n$ matrices over a field F and let W_0 be the subspace spanned by the matrices C of the form $C := AB - BA$. Recall that the trace of an $n\times n$ matrix is equal to the sum of the n entries in the diagonal and let W_1 denote the nullspace of the trace function on W.

(1) [5] Show that $W_0 \subseteq W_1$.
(2) [5] Construct in W_0 a linearly independent set of n^2-1 elements.
(3) [5] Deduce that W_0 is exactly the subspace of matrices which have trace zero.

[Use the fact that the trace $tr: W \longrightarrow F$ is a linear functional]

SOLUTION

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(1) Let $\begin{array}{cccc} \text{tr}: & W & \longrightarrow & F \\ & A & \mapsto & \text{tr}(A) \end{array}$ be the trace function on W.

•• Let $A = \left(a_{ij}\right)$, $B = \left(b_{ij}\right) \in W$, then: $\text{tr}(AB) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}a_{ik} = \text{tr}(BA)$.

•• Since tr is linear, then $\text{tr}(AB-BA) = \text{tr}(AB) - \text{tr}(BA)$.

• Let $C := AB - BA$. Then, $\text{tr}(C) = \text{tr}(AB-BA) = \text{tr}(AB) - \text{tr}(BA) = 0$. That is, $C \in W_1$.

Therefore $W_0 \subseteq W_1$.

(2) Let E_{ij} be the $n\times n$ matrix, which takes 1 in the ith row & jth column and zero elsewhere.

•• For every i, j, k, l, with $k \neq l$, we have $\left\langle \begin{array}{rcl} E_{ik}E_{kj} & = & E_{ij} \\ E_{ik}E_{lj} & = & 0 \end{array} \right.$.

•• Let $\left\langle \begin{array}{l} \text{for each } \; i \neq 1\,,\ A_{ii} := E_{ii} - E_{11} = E_{i1}E_{1i} - E_{1i}E_{i1} \in W_0 \\ \text{for each } \; i \neq j\,,\ A_{ij} := E_{ij} = \underbrace{E_{i1}E_{1j}}_{=E_{ij}} - \underbrace{E_{1j}E_{i1}}_{=0} \in W_0 \end{array} \right.$; $\left|\left\{A_{ij}\right\}\right| = n^2 - 1$.

• $\left\{A_{ij}\right\}$ Linearly Independent :

$$\sum_{i\neq j} c_{ij}A_{ij} + \sum_{i\neq 1} c_i A_{ii} = 0 \quad ; \quad \sum_{i\neq j} c_{ij}E_{ij} + \sum_{i\neq 1} c_i\left(E_{ii} - E_{11}\right) = 0 \quad ; \quad \left\langle \begin{array}{l} c_{ij} = 0\,,\ \forall\, i \neq j \\ c_i = 0\,,\ \forall\, i \neq 1 \end{array} \right. \quad ; \quad \left(-\sum_{i\neq 1} c_i = 0 \text{ is redundant}\right)$$

(3) • Since tr is a linear functional, then W_1 is a hyperspace; that is, $\dim(W_1) = \dim(W) - 1 = n^2 - 1$.

• Since $W_0 \subseteq W_1$, then $\dim(W_0) \leq n^2 - 1$.

•• (2) $\Longrightarrow \dim(W_0) \geq n^2 - 1$ and so $\dim(W_0) = n^2 - 1$.

- $$\left.\begin{array}{r} W_0 \subseteq W_1 \\ \dim(W_0) = \dim(W_1) \end{array}\right\} \Longrightarrow W_0 = W_1.$$

Problem 3. [20] [Sections 6.2-6.4, 7.1]

Find the characteristic and minimal polynomials for each one of the following matrices, and indicate whether the matrix is diagonalizable or not with justification.

(1) [6] $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ over $\mathbb{R}$

(2) [8] $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ over $\mathbb{R}$

(3) [6] $A = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ over $\mathbb{C}$.

SOLUTION

Let f and p denote the characteristic and minimal polynomials, respectively.

(1) •• $f = (x-1)^2(x-2)$ over $\mathbb{R}$.

•• $A \neq I$, $A \neq 2I$, and $(A-I)(A-2I) = 0$, so $p = (x-1)(x-2)$.

•• A is diagonalizable since p is a product of distinct linear factors (Section 6.4 - Theorem 6).

(or rank$(A-I) = 1$ so that nullity$(A-I) = 2$ and hence A is diagonalizable (Section 6.2 - Theorem 2))

(2) •• $f = (x-1)(x^2-x+1)$ over $\mathbb{R}$.

•• $x^2 - x + 1$ is a prime factor on $\mathbb{R}$ (since no roots).

•• By Generalized Cayley-Hamilton Theorem, $p = f = (x-1)(x^2-x+1)$.

•• A is NOT diagonalizable as p is NOT a product of (distinct) linear factors (Section 6.4 - Theorem 6).

(3) Direct computation or using the fact that A has a companion form, we get:

•• $f = p = x^4 + x^2 - 2$.

•• $f = p = (x^2-1)(x^2+2) = (x-1)(x+1)(x-i\sqrt{2})(x+i\sqrt{2})$ over $\mathbb{C}$.

•• A is diagonalizable since it has 4 distinct characteristic values.

Problem 4. **[10]** [**Sections 6.2-6.5**]

Consider the real matrices $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$, $r \neq 0$.

(1) **[5] Explain** why there must exist a 2×2 invertible matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal.

(2) **[5] Find** all matrices P satisfying (1).

SOLUTION

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(1) •• Charac. Poly.$(A) = x(x-2)$; Two distinct charac. values: $0, 2$; A diagonalizable.

•• Charac. Poly.$(A) = (x+r-1)(x-r-1)$; Two distinct charac. values: -r+1, r+1 ; B diagonalizable.

• Conclusion: Since A and B are diagonalizable and $AB = BA$, then there exists an invertible matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal (Theorem 6.5-8).

(2) Direct :

Let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc \neq 0$.

•••• $P^{-1}AP \begin{cases} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \; ; \; c = -a \;\&\; d = b \; ; \; P = \begin{pmatrix} a & b \\ -a & b \end{pmatrix} \; ; \; P^{-1}BP = \begin{pmatrix} -r+1 & 0 \\ 0 & r+1 \end{pmatrix} \\ = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \; ; \; c = a \;\&\; d = -b \; ; \; P = \begin{pmatrix} a & b \\ a & -b \end{pmatrix} \; ; \; P^{-1}BP = \begin{pmatrix} r+1 & 0 \\ 0 & -r+1 \end{pmatrix} \end{cases}$

• Conclusion: $P = \begin{pmatrix} a & b \\ -a & b \end{pmatrix}$ or $\begin{pmatrix} a & b \\ a & -b \end{pmatrix}$, for any nonzero $a, b \in \mathbb{R}$.

Through Characteristic Vectors :

•• Matrix A : Charac. values $\begin{cases} \lambda = 0 \; ; \; \text{charac. vectors} = \begin{pmatrix} a \\ -a \end{pmatrix}, \text{ for any nonzero } a \in \mathbb{R} \\ \lambda = 2 \; ; \; \text{charac. vectors} = \begin{pmatrix} b \\ b \end{pmatrix}, \text{ for any nonzero } b \in \mathbb{R} \end{cases}$

•• Matrix B : Charac. values $\begin{cases} \lambda = -r+1 \; ; \; \text{charac. vectors} = \begin{pmatrix} a \\ -a \end{pmatrix}, \text{ for any nonzero } a \in \mathbb{R} \\ \lambda = r+1 \; ; \; \text{charac. vectors} = \begin{pmatrix} b \\ b \end{pmatrix}, \text{ for any nonzero } b \in \mathbb{R} \end{cases}$

• So, $P = \begin{pmatrix} a & b \\ -a & b \end{pmatrix}$ or $\begin{pmatrix} b & a \\ b & -a \end{pmatrix}$, for any nonzero $a, b \in \mathbb{R}$.

Problem 5. [25] [**Sections 7.1-7.4**]

Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(1) **[2]** Reduce $xI - A$ to its Smith normal form.

(2) **[4]** Use the Smith normal form of A to find its invariant factors.

(3) **[3]** Find the rational form for A.

(4) **[8]** Find the Jordan form for A.

(5) **[8]** Let T be a linear operator on $\mathbb{R}^4$ such that A is the matrix associated to T in the standard basis $\{e_1, e_2, e_3, e_4\}$.

(a) Find the respective T-annihilators of e_1, e_2, e_3, and e_4.

(b) Announce the Cyclic Decomposition Theorem for this case.

(c) Find an explicit cyclic decomposition of $\mathbb{R}^4$ under T.

SOLUTION

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(1) •• $xI - A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x^2(x-1) \end{pmatrix}$; $f_1 = 1$ / $f_2 = 1$ / $f_3 = x-1$ / $f_4 = x^2(x-1)$.

(2) By the *Invariant Factors Theorem* combined with the *uniqueness of the Smith normal form*, we get

$xI - A \sim \begin{pmatrix} x^2(x-1) & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, where the invariant factors are

$$\begin{cases} \bullet\bullet\ p_1 = x^2(x-1), & \text{Minimal Polynomial} \\ \bullet\bullet\ p_2 = x-1 \end{cases}$$

(3)

$$\begin{cases} \bullet\ \text{Charac. Poly. } f = p_1 p_2 = x^2(x-1)^2 \\ p_1 = x^2(x-1) = x^3 - x^2 \\ p_2 = x - 1 \end{cases}$$

•• Rational Form : $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(4) $f = p_1 p_2 = x^2(x-1)^2$

Characteristic Values $\begin{cases} \bullet\ c_1 = 0 & \text{with } d_1 = 2, r_1 = 2 \\ \bullet\ c_2 = 1 & \text{with } d_2 = 2, r_2 = 1 \end{cases}$

$\Rightarrow A \sim \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, where $\begin{cases} \bullet\ A_1 & 2\times 2 \text{ matrix (since } d_1 = 2) \\ \bullet\ A_2 & 2\times 2 \text{ matrix (since } d_2 = 2) \end{cases}$

For $c_1 = 0$:

- $J_1^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ since $r_1 = 2$
- $\Rightarrow A_1 = J_1^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

For $c_2 = 1$:

- $J_1^{(2)} = \begin{pmatrix} 1 \end{pmatrix}$ since $r_2 = 1$
- $\Rightarrow A_2 = \begin{pmatrix} J_1^{(2)} & 0 \\ 0 & J_2^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ since $d_2 = 2$ forces $J_2^{(2)} = \begin{pmatrix} 1 \end{pmatrix}$

Jordan Form : $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(5) (a) The T-annihilator p_α of any vector α satisfies $\begin{cases} p_\alpha \text{ divides } p_1 = x^2(x-1) \\ p_\alpha \text{ is minimal with } (p_\alpha T)\alpha = 0 \end{cases}$

- $Te_1 = e_1 \Rightarrow p_{e_1} = x - 1$.
- $Te_3 = e_3 \Rightarrow p_{e_3} = x - 1$.
- $T^2 e_2 = Te_2 = e_3 \Rightarrow p_{e_2} = x(x-1)$.
- $\{e_4, Te_4 = e_2, T^2 e_4 = e_3\}$ linearly independent $\Rightarrow p_{e_4} = p_1 = x^2(x-1)$.

(b) Cyclic Decomposition Theorem: $\exists\, \alpha, \beta \in \mathbb{R}^4$ such that
$\begin{cases} \bullet\ \mathbb{R}^4 = Z(\alpha, T) \oplus Z(\beta, T) \\ \bullet\ \text{with } p_\alpha = p_1 \text{ and } p_\beta = p_2 \end{cases}$

(c) Therefore, $\begin{cases} \bullet\ \alpha = e_4 \\ \bullet\ \beta = e_1 \end{cases}$; that is, $\begin{cases} \mathbb{R}^4 = Z(e_4, T) \oplus Z(e_1, T)) \\ \text{with } Z(e_4, T) = \mathbb{R}e_4 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3 \\ \text{and } Z(e_1, T) = \mathbb{R}e_1 \end{cases}$

Problem 6. **[15]** **[Section 8.3 - Ex 11]**

Let V be a finite-dimensional inner product space.

(1) **[3]** Prove that any orthogonal projection of V on a subspace W is self-adjoint.

(2) **[8]** Let E be a projection (i.e., $E = E^2$). Prove that if E is normal, then E and $E^\star$ have the same nullspace, and use this fact to show that $V = \text{Range}(E) \bigoplus (\text{Range}(E))^\perp$

(3) **[4]** Deduce from (1) and (2), that "*a projection is normal if and only if it is self-adjoint*."

SOLUTION

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(1) ••• Let $\alpha, \beta \in V$ and let $E : V \longrightarrow W$ be the orthogonal projection. Then:

$$\begin{aligned}
(E\alpha \mid \beta) &= (E\alpha \mid \beta - E\beta + E\beta) \\
&= \underbrace{(E\alpha \mid \beta - E\beta)}_{=0} + (E\alpha \mid E\beta) \;=\; (E\alpha \mid E\beta) \\
&= (E\alpha - \alpha + \alpha \mid E\beta) \\
&= \underbrace{(E\alpha - \alpha \mid E\beta)}_{=0} + (\alpha \mid E\beta) \;=\; (\alpha \mid E\beta)
\end{aligned}$$

So, $E^\star = E$.

(2) $\text{Nullspace}(E) = \text{Nullspace}(E^\star)$ **:**

• . E is normal, then $EE^\star = E^\star E$.

•• . $\|E\alpha\|^2 = (E\alpha \mid E\alpha) = (\alpha \mid E^\star E\alpha) = (\alpha \mid EE^\star\alpha) = (E^\star\alpha \mid E^\star\alpha) = \left\|E^\star\alpha\right\|^2$.

• . $E\alpha = 0 \Longleftrightarrow E^\star\alpha = 0$

$V = \text{Range}(E) \bigoplus (\text{Range}(E))^\perp$ **:**

•• . $\text{Nullspace}(E^\star) = (\text{Range}(E))^\perp$ since $E^\star\alpha = 0 \Longleftrightarrow (E\beta \mid \alpha) = (\beta \mid E^\star\alpha) = 0\,,\ \forall\, \beta \in V$

•• . $V = \text{Range}(E) \bigoplus \text{Nullspace}(E) = \text{Range}(E) \bigoplus \text{Nullspace}(E^\star) = \text{Range}(E) \bigoplus (\text{Range}(E))^\perp$

••• **(3)** ($\Longrightarrow$) Assume E is a normal projection. By (2), $V = \text{Range}(E) \bigoplus (\text{Range}(E))^\perp$. Hence, E is orthogonal on $\text{Range}(E)$ and so self-adjoint by (1).

• ($\Longleftarrow$) Trivial.