Comprehensive Exam, Math 551

Q1: **(a)** Let $P \in \operatorname{Spec}(R)$ and consider the homomorphism

$$\varphi : R[x] \longrightarrow R/_P[x] \; ; \quad f(x) = a_0 + a_1 x + \cdots + a_n x^n$$
$$f(x) \longmapsto \overline{f}(x) \qquad \overline{f}(x) = \overline{a}_0 + \overline{a}_1 x + \cdots + \overline{a}_n x^n \pmod{P}$$

Clearly φ is onto and $\operatorname{Ker}\varphi = P[x]$.

Therefore $\dfrac{R[x]}{P[x]} \simeq R/_P[x]$ is an integral domain and so

$P[x]$ is a prime ideal of $R[x]$.

(b) $M \in \operatorname{Max}(R)$, by (a) $\dfrac{R[x]}{M[x]} \simeq R/_M[x]$ is not a field,

So $M[x]$ is not a maximal ideal of $R[x]$.

(c) Q P-primary $\Longrightarrow$ $Q[x]$ $P[x]$-primary

Let $f(x) = \sum_{j=j_0}^{n} a_j x^j$ and $g(x) = \sum_{k=k_0}^{s} b_k x^k$ such that

$f(x)g(x) \in Q[x]$ and $f(x) \notin P[x]$ with $a_n \neq 0, a_n \notin P$

(for if $a_n \in P$, then $f(x) = \sum_{j=j_0}^{n-1} a_j x^j + a_n x^n$, we replace $f(x)$ by $\sum_{j=j_0}^{n-1} a_j x^j$) We proceed by induction on $s - k_0$ = Number of terms in $g(x)$

For $s - k_0 = 0$, (ie $g(x) = b_{k_0} x^{k_0} = b_s x^s$, we do have

$f(x)g(x) = \sum_{j=j_0}^{n} b_s a_j x^{s+j} \in Q[x] \Rightarrow b_s a_j \in Q \;\forall j$.

As $a_n \notin P$ and $b_s a_n \in Q$, Then $b_s \in Q$. So $g(x) \in Q[x]$

We assume that $s - k_0 \geq 1$ and the induction hypothesis.

First, since $b_s a_n \in Q$ and $a_n \notin P$, then $\boxed{b_s \in Q}$

Let t be the smallest positive integer such that $\boxed{b_s^t \in P}$

Then $b_s^{t-1} \notin I$ and so $b_s^{t-1} f(x) = h(x) \notin P[x]$ $(a_n b_s^{t-1} \notin I$

We do have: $h(x) g(x) = b_s^{t-1} f(x) g(x) \in I[x]$

Also $b_s x^s \cdot b_s^{t-1} f(x) = b_s^t x^s f(x) \in I[x]$ (since $b_s^t \in I$)

Then: $(g(x) - b_s x^s) b_s^{t-1} f(x) = \underbrace{b_s^{t-1} f(x) g(x)}_{\in I[x]} - \underbrace{b_s^t x^s f(x)}_{\in I[x]} \in I[x]$

As the number of terms in $g(x) - b_s x^s$ is less than s-k_0 by induction hypothesis, $g(x) - b_s x^s \in Q[x]$.

But $b_s x^s \in Q[x]$ (since $b_s \in Q$) and so $g(x) \in Q[x]$ ∎

(d) $I = \langle 4x^2 - 4, 2x^3 - 2x, x^4 - x^2 \rangle$

$I = \langle 4(x^2-1), 2x(x^2-1), x^2(x^2-1) \rangle = \langle x^2-1 \rangle \langle 4, 2x, x^2 \rangle$

$I = \langle x-1 \rangle \langle x+1 \rangle \langle x, 2 \rangle^2 = Q_1 \cap Q_2 \cap Q_3$

$Q_1 = \langle x-1 \rangle$ and $Q_2 = \langle x+1 \rangle$ are prime ideals.

$Q_3 = \langle x, 2 \rangle^2$ is M-primary ideal where $M = \langle x, 2 \rangle$.

Q2.

$$\begin{tikzcd}
& \mathrm{Ker}(\alpha) \arrow[r] \arrow[d, hook] & \mathrm{Ker}\,\beta \arrow[r] \arrow[d, hook] & \mathrm{Ker}\,\gamma \arrow[d, hook] \arrow[dddll, bend left, red] & \\
& L_1 \arrow[r, "f_1"] \arrow[d, "\alpha"] & M_1 \arrow[r, "g_1"] \arrow[d, "\beta"] & N_1 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & L_2 \arrow[r, "f_2"] \arrow[d] & M_2 \arrow[r, "g_2"] \arrow[d] & N_2 \arrow[d] & \\
& \mathrm{Coker}(\alpha) \arrow[r] & \mathrm{Coker}(\beta) \arrow[r] & \mathrm{Coker}(\gamma) &
\end{tikzcd}$$

By Snake Lemma, the sequence is Exact.

ⓐ α and γ are monomorphisms, that is, $\mathrm{Ker}\,\alpha = \mathrm{Ker}\,\gamma = 0$

So $0 \longrightarrow \mathrm{Ker}\,\beta \longrightarrow 0 \Longrightarrow \mathrm{Ker}\,\beta = 0 \Rightarrow \beta$ monomorphism

ⓑ α and γ epimorphisms, that is, $\mathrm{Coker}\,\alpha = \mathrm{Coker}\,\gamma = 0$

So $0 \longrightarrow \mathrm{Coker}\,\beta \longrightarrow 0 \Rightarrow \mathrm{CorKer}\,\beta = 0 \Rightarrow \beta$ epimorphism.

ⓒ α is epimorphism and β monomorphism $\Rightarrow \gamma$ monomorphism.

Let $y \in N_1$ such that $\gamma(y) = 0$.

- g_1 is surjective $\Rightarrow \exists\, x \in M_1$ such that $g_1(x) = y$.

So $\gamma(g_1(x)) = \gamma(y) = 0$ and thus, $g_2(\beta(x)) = \gamma(g_1(x)) = 0$

Then $\beta(x) \in \mathrm{Ker}\,g_2 = \mathrm{Im}\,f_2$. So there is $z \in L_2$ such that $\beta(x) = f_2(z)$.

- α is surjective, then $\exists\, t \in L_1$ such that $\alpha(t) = z$.

So $\beta(x) = f_2(z) = f_2(\alpha(t)) = \beta(f_1(t))$. But since

- β is monomorphism, $x = f_1(t)$. So

$y = g_1(x) = g_1(f_1(t)) = 0$ since $g_1 f_1 = 0$.

Thus $\mathrm{Ker}\,\gamma = 0$ and therefore γ monomorphism.

(d) β epimorphism and γ monomorphism $\Longrightarrow$ α epimorphism:

Let $y \in L_2$. Then $f_2(y) \in M_2 = \operatorname{Im}\beta$, β surjective.

So $f_2(y) = \beta(x)$ for some $x \in M_1$.

Next, $0 = \underbrace{g_2(f_2(y))}_{g_2 \circ f_2 = 0} = g_2(\underbrace{\beta(x)) = \gamma(}_{g_2 \circ \beta = \gamma \circ g_1}g_1(x))$

So $g_1(x) \in \operatorname{Ker}\gamma = 0$ since γ is a monomorphism.

Then $x \in \operatorname{Ker} g_1 = \operatorname{Im} f_1$. So there is $z \in L_1$ such that $x = f_1(z)$.

Now, $f_2(y) = \beta(x) = \beta\underbrace{(f_1(z)) = f_2(}_{\beta \circ f_1 = f_2 \circ \alpha}\alpha(z))$

But since f_2 is injective, $y = \alpha(z)$, and therefore α is surjective, as desired. $\square$

Q3 R an integral Domain.

(a) Every injective R-module is divisible.

Let M be an injective R-module, $m \in M$ and $0 \neq r \in R$.

Consider the R-map: $f: I = Rr \longrightarrow M$. ($I$ ideal of R)
$ar \longmapsto am$

By Baer Criterion, there is $g: R \longrightarrow M$ R-map s.t. $g|_I = f$.

Now $m = g(r) = rg(1)$ and $g(1) = x \in M$ with $m = rx$. Thus m is divisible by r and so M is Divisible.

(b) Assume that R is a PID and M is Divisible.

Again, we will apply Baer Criterion.

Let $0 \neq I$ be an ideal of R and $f: I \longrightarrow M$ R-map.

- Since R is a PID, $I = Ra$ for some $0 \neq a \in R$.
- $f(a) \in M$, $a \neq 0$, and M divisible $\Longrightarrow f(a) = ax$ for some $x \in M$.
- Now, let $g: R \longrightarrow M$
  $r \longmapsto rx$.
- Clearly g is an R-map and $g|_I = f$ since $\forall\, ra \in I$
  $g(ra) = (ra)x = r(ax) = rf(a) = f(ra)$.

Therefore M is injective (by Baer Criterion).

(c) Follows immediately from the fact the product and the Sum of divisible modules are divisible modules.

- As an abelian group, $\mathbb{Q}$ is a $\mathbb{Z}$-module which is divisible. So $\prod \mathbb{Q}$ and $\coprod \mathbb{Q}$ are divisible modules over $\mathbb{Z}$.
- $\mathbb{Z}$ is a PID $\Longrightarrow \prod \mathbb{Q}$ and $\coprod \mathbb{Q}$ are injective $\mathbb{Z}$-modules by (b)

d) $\mathbb{Z}_{p^\infty}$, as an abelian group, is a divisible $\mathbb{Z}$-module

Then $\bigoplus_{p\in\mathbb{P}} \mathbb{Z}_{p^\infty}$ is a divisible $\mathbb{Z}$-module (as a direct sum of divisible $\mathbb{Z}$-module). As $\mathbb{Z}$ is a PID, by (b)

$\bigoplus_{p\in\mathbb{P}} \mathbb{Z}_{p^\infty}$ is injective.

Q4 $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ exact sequence of (left) R-modules.

ⓐ Assume that $M \simeq L \oplus N$ and M injective. Then L and N are injective.

- Follows immediately from the fact every summand D of an injective module E is injective.
- A direct proof is: consider the diagram of R-maps

$$
\begin{tikzcd}
 & L \arrow[r, hook, shift left, "i"] & M = L \oplus N \arrow[l, shift left, "p"] \\
0 \arrow[r] & A \arrow[u, "f"] \arrow[r, "u"] & B
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
 & M & \\
0 \arrow[r] & A \arrow[u, "i\circ f"] \arrow[r, "u"] & B \arrow[ul, dashed, "g"']
\end{tikzcd}
$$

$i : L \longrightarrow M = L \oplus N$
$\quad x \longmapsto x$

$p : M = L \oplus N \longrightarrow L$
$\quad x + y \longmapsto x.$

Since M is injective, there is $g : B \longrightarrow M$ s.t. $i \circ f = g \circ u$

Now consider: $\bar{g} : B \longrightarrow L$ defined by $\bar{g}(x) = p(g(x))$.

We do have: For every $a \in A$, $f(a) \in L$ and so:

$$(\bar{g} \circ u)(a) = \bar{g}(u(a)) = p(g(u(a))) = p(i \circ f(a)) = p(f(a)) = f(a)$$

Thus $\bar{g} \circ u = f$, as desired. Therefore L is injective

Similarly, N is injective by interchanging their Role.

ⓑ Assume that L is injective. We show that the sequence splits

Consider the diagram: $\longrightarrow$

$$
\begin{tikzcd}
 & L & \\
0 \arrow[r] & L \arrow[u, "id_L"] \arrow[r, "f"] & M \arrow[ul, dashed, "g"']
\end{tikzcd}
$$

Since L is injective, there is $g : M \longrightarrow L$ Such that $id_L = g \circ f$, as desired ∎

(c) Assume ${}_RL$ and ${}_RM$ are injective. By (b), the given exact sequence splits, i.e. $M \simeq L \oplus N$. By (a), ${}_RN$ is injective.

(d) Not necessarily.
Counter example: Consider the following exact sequence of Abelian groups ($\mathbb{Z}$-modules):

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Since $\mathbb{Z}$ is a PID, a $\mathbb{Z}$-module X is injective if and only if X is a divisible Abelian group. It follows that the divisible Abelian groups $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules. However, $\mathbb{Z}$ is clearly not divisible (e.g. the equation $2x = 3$ has no solution in $\mathbb{Z}$), whence ${}_{\mathbb{Z}}\mathbb{Z}$ is not injective.

Q5 Let R be an Artinian commutative ring

(a) R a domain $\Rightarrow R = K = qf(R)$ is a field.

Let $0 \neq a \in R$ and consider the chain (descending) of ideals $Ra \supseteq Ra^2 \supseteq \dots \supseteq Ra^n \supseteq Ra^{n+1} \dots$

Since R is Artinian, there is $n_0 \geq 1$ s.t. $Ra^{n_0} = Ra^n$ $\forall n \geq n_0$.

In particular $Ra^{n_0} = Ra^{n_0+1}$. So $a^{n_0} \in Ra^{n_0+1}$, that is $a^{n_0} = \alpha a^{n_0+1}$ for some $\alpha \in R$. Thus $a^{n_0}(1-\alpha a) = 0$.

As R is an integral domain, $a^{n_0} \neq 0$ and so $1 - \alpha a = 0$.

Hence $1 = \alpha a$ and so $a \in U(R)$. Therefore R is a field.

(b) Every ~~maximal~~ prime ideal is a maximal ideal.

Let $P \in \mathrm{Spec}(R)$. Then R/P is an Artinian integral domain, and so by a) R/P is a field. Thus P is a maximal ideal of R.

(c) R has only finitely many prime ideals.

Consider $\Sigma = \{ \bigcap_{i=1}^{n} m_i \mid m_i \in \mathrm{Max}(R) \}$ the set of all finite intersections of prime = maximal ideals of R. Since $\Sigma \neq \phi$ and R is Artinian, Σ has a minimal element. Say $m_1 \cap \dots \cap m_{n_0}$. Next, let $m \in \mathrm{Max}(R)$, then

$m \cap m_1 \cap \dots \cap m_{n_0} \in \Sigma$ and $m \cap m_1 \cap \dots \cap m_{n_0} \subseteq m_1 \cap \dots \cap m_{n_0}$.

By minimality, $m \cap m_1 \cap \dots \cap m_{n_0} = m_1 \cap \dots \cap m_{n_0}$.

So $m = m_{j_0}$ for some $j_0 = 1, \dots n_0$. Hence $\mathrm{Max}(R) = \mathrm{Spec}(R) = \{m_1, \dots, m_{n_0}\}$, as desired.

(d) $N = \text{Nilradical}(R) = \bigcap_{i=1}^{n} M_i$, $\{M_1, \ldots, M_n\} = \text{Max}(R) = \text{Spec}(R)$.

Then $N^k = 0$ for some $k \geq 1$ (ie N Nilpotent).

By way of contradiction, Suppose that $\forall k \geq 1$ $N^k \neq 0$.

Consider the (descending) chain of nonzero ideals

$N \supseteq N^2 \supseteq \cdots\cdots$ Then there is $k_0 \geq 1$ (smallest integer)

Such that $N^{k_0} = N^n$ $\forall n \geq k_0$.

Let $\Sigma = \{ I$ ideal of R s.t. $IN^{k_0} \neq 0 \}$.

- clearly $N \in \Sigma$ (as $N^{k+1} \neq 0$) and so $\Sigma \neq \phi$.
- Let I_0 be a minimal element of Σ. then $I_0 N^{k_0} \neq 0$.

So there is $x \in I_0$ such that $xN^{k_0} \neq 0$, $x \neq 0$.

Then the ideal $Rx \subseteq I_0$ and by minimality

$Rx = I_0$.

- $(xN^{k_0})N^{k_0} = xN^{2k_0} = xN^{k_0} \neq 0$, and $xN^{k_0} \subseteq I_0 = Rx$

Again by minimality, $xN^{k_0} = Rx$. So $x = xy$ for some

$y \in N^{k_0} \subseteq N$, but $x(1-y) = 0 \implies x = 0$ since $1-y \in U(R)$

a contradiction. Hence $N^k = 0$ for some $k \geq 1$ ∎

Q6, R a ring, $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ Exact

(a) M Noetherian $\Longrightarrow$ L & N Noetherian

Follows immediately from the fact that submodules and quotient modules of Noetherian modules are Noetherian

So
- $L \simeq f(L)$, as a submodule of M is Noetherian
- $N \simeq M/\operatorname{Ker} g$, as a quotient of Noeth. mod. is Noetherian

(b) L and N are Noetherian $\Longrightarrow$ M Noetherian.

Let $M_1 \subseteq M_2 \subseteq \dots$ be a chain of submodules of M. Then $f^{-1}(M_1) \subseteq f^{-1}(M_2) \subseteq \dots \subseteq f^{-1}(M_n) \subseteq \dots$ is a chain of submodules of L and $g(M_1) \subseteq g(M_2) \subseteq \dots$ is a chain of submodules of N. As L and N are Noetherian, the two chains stabilize that is, $\exists\, n_1 \geq 1$ s.t. $f^{-1}(M_k) = f^{-1}(M_{k+1}), \forall k \geq n_1$ and there is $n_2 \geq 1$ s.t. $g(M_k) = g(M_{k+1}), \forall k \geq n_2$.

- let $n = \max(n_1, n_2)$, then $\forall k \geq n$, $f^{-1}(M_k) = f^{-1}(M_{k+1})$ and $g(M_k) = g(M_{k+1})$.
- Next consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & f^{-1}(M_k) \arrow[r, "f"] \arrow[d, "\parallel"'] & M_k \arrow[r, "g"] \arrow[d, "\alpha"] & g(M_k) \arrow[r] \arrow[d, "\parallel"] & 0 \\
0 \arrow[r] & f^{-1}(M_{k+1}) \arrow[r, "f"] & M_{k+1} \arrow[r, "g"] & g(M_{k+1}) \arrow[r] & 0
\end{tikzcd}$$

By the five Lemma, $\alpha: M_k \longrightarrow M_{k+1}$ is an isomorphism but since $M_k \subseteq M_{k+1}$, $M_k = M_{k+1}$ $\forall k \geq n$ and so M is Noetherian.

(c) M Artinian $\Longrightarrow$ L and N Artinian.

Similar to (a) Since submodules and quotients of Artinian modules are Artinian modules.

So $L \simeq f(L)$ as a submodule of M is Artinian.

and $N \simeq M/\ker g$ as a quotient module of M is Artinian.

(d) L and N are Artinian $\Longrightarrow$ M Artinian.

Similar to the proof of Noetherian case.

Part II: Solve each of the following two questions:

Q7. (8 points) Compute the following Abelian groups (*up to isomorphism*):

(a) $\mathbb{Z}_{12} \otimes_{\mathbb{Z}} \mathbb{Z}_{30} \simeq \mathbb{Z}_{\gcd(12,30)} = \mathbb{Z}_6$.

(b) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \simeq \mathbb{Q}$.

Q8. (12 points) Prove or disprove (showing full details):

(a) Every commutative Noetherian ring R with zero Jacobson radical is semisimple.

FALSE
Counter example: Consider $R = \mathbb{Z}$. The ring of integers is commutative, $J(\mathbb{Z}) = 0$ and $\mathbb{Z}$ is Noetherian. Since $\mathbb{Z}$ is not Artinian, $\mathbb{Z}$ is not semisimple.

(b) Every left semisimple ring is right semisimple.

TRUE
Proof:

By the **Wedderburn–Artin Theorem:** A ring R is left semisimple if and only if R is isomorphic to a direct product of finitely many finite matrix rings over division rings.

Let R be left semisimple. Then there exist finitely many division rings $D_1, \cdots, D_k$ and positive integers $n_1, \cdots, n_k$ such that

$$R \simeq M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k).$$

It follows that

$$\begin{aligned} R^{op} &\simeq (M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k))^{op} \\ &\simeq M_{n_1}(D_1)^{op} \times \cdots \times M_{n_k}(D_k)^{op} \\ &\simeq M_{n_1}({D_1}^{op}) \times \cdots \times M_{n_k}({D_k}^{op}). \end{aligned}$$

Since $D_1^{op}, \cdots, D_k^{op}$ are also division rings, R^{op} is left semisimple whence R is right semisimple (R and R^{op} are anti-isomorphic).

(c) If R is a Noetherian commutative ring, then every proper ideal of R has a unique primary decomposition.

FALSE
Counter example: Consider the commutative Noetherian ring $R := \mathbb{R}[x, y]$ and the ideal $I := (x^2, xy)$. Then I has two different *minimal* primary decompositions

$$(x^2, xy) = (x) \cap (x, y)^2 = (x) \cap (x^2, y).$$

Notice that

- $\mathbb{R}[x, y]/(x) \simeq \mathbb{R}[y]$ (an integral domain), whence (x) is a prime ideal of R.
- $\mathbb{R}[x, y]/(x, y) \simeq \mathbb{R}$ (a filed), whence (x, y) is a maximal ideal of R.
- It is clear that $\sqrt{(x, y)^2} = (x, y) = \sqrt{(x^2, y)}$, whence both $(x, y)^2$ and (x^2, y) are (x, y)-primary (any ideal with maximal radical is primary).
- The set of associated primes of I is $\{(x), (x, y)\}$.

GOOD LUCK