King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
ODE Comprehensive Exam
The Second Semester of 2020-2021 (202)
Time Allowed: 120mn

Name: **ID number:**

This is a closed book exam

Problem #	Marks	Maximum Marks
1		25
2		25
3		25
4		25
5		25
Total		$25 \times 4 = 100$

Solve only 4 problems of your choice.

Remark: In case a student solves all 5 problems, only the first 4 on the exam sheets will be graded.

Problem 1:(25pts)

1.)(12pts) Find the explicit solution of the IVP. Give the largest interval of definition of the solution.

$$\frac{dy}{dx} = (y^2-1)x, \quad x, y \in \mathbb{R},$$
$$y(0) = y_0.$$

2.)(13pts) Show that the IVP has a unique solution in some interval around $x = 0$.

$$\frac{dy}{dx} = -\frac{1}{(y-x)^2},$$
$$y(0) = 1.$$

<u>Solution</u>

1) • If $y_0=-1$, then $y=-1$ is the solution on $(-\infty,\infty)$

• If $y_0=1$, then $y=1$ is the solution on $(-\infty,\infty)$

• If $y_0\neq\pm1$, then $\int \frac{dy}{y^2-1}=\int dx \Rightarrow \ln\left|\frac{y-1}{y+1}\right| = x^2+C$

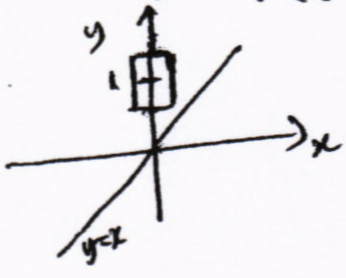
$\frac{y-1}{y+1}=ce^{x^2} \Rightarrow y=\frac{1+ce^{x^2}}{1-ce^{x^2}}$ and $c=\frac{y_0-1}{y_0+1}$

• If $y_0\in(-1,1)$, $c<0$ and $y=\frac{1+ce^{x^2}}{1-ce^{x^2}}$, $x\in(-\infty,\infty)$

• If $y_0\in(-\infty,-1)$, $c>1$ and $y=\frac{1+ce^{x^2}}{1-ce^{x^2}}$, $x\in(-\infty,\infty)$

• If $y_0\in(1,\infty)$, $0<c<1$ and $y=\frac{1+ce^{x^2}}{1-ce^{x^2}}$, $x\in\left(-\sqrt{-\ln c}, \sqrt{-\ln c}\right)$

2) $f(x,y) = -\frac{1}{(y-x)^2}$



$D=\{(x,y) / -\frac{1}{4}\le x\le\frac{1}{4}, \ \frac{1}{2}\le y\le\frac{3}{2}\}$

$\frac{\partial f}{\partial y}=\frac{2}{(y-x)^3}$

$|f(x,y)| = \frac{1}{(y-x)^2}$, $\frac{1}{4}\le y-x\le\frac{7}{4}$, $\frac{1}{16}\le(y-x)^2\le\frac{47}{16}$

$\frac{1}{64}\le(y-x)^3\le\frac{329}{64}$

Thus, $|f(x,y)|\le 16$

$\left|\frac{\partial f}{\partial y}(x,y)\right|\le 128$, $f, \frac{\partial f}{\partial y}$ are continuous on D.

$\Rightarrow$ The IVP has a unique solution on $I=\left[-\frac{1}{36}, \frac{1}{36}\right]$

Problem 2:(25pts) Consider the system

$$\frac{dx}{dt} = y(x^2+y^2),$$
$$\frac{dy}{dt} = -x(x^2+y^2),$$
$$\frac{dz}{dt} = z - x^2 - y^2.$$

1.)(4pts) Verify that $X_1(t) = (\sin t, \cos t, 1)$ and $X_2(t) = (0, 0, e^t)$ are solutions of the system.
2.)(10pts) Write the linearized system at the periodic solution $X_1(t)$.
3.)(6pts) Find all characteristics multipliers of the linearized system at $X_1(t)$.
4.)(5pts) Deduce the stability of the periodic solution $X_1(t)$.

Solution

1) $1-\sin^2 t-\cos^2 t = 0$, $(\sin t)' = \cos t$, $(\cos t)' = -\sin t$

$\Rightarrow$ $x_1(t)$ is a solution to the system.

it is also clear that $x_2(t)$ is another solution

2) $$J = \begin{pmatrix} 2xy & x^2+3y^2 & 0 \\ -3x^2-y^2 & -2xy & 0 \\ -2x & -2y & 1 \end{pmatrix}, \quad J(\sin t, \cos t, 1) = \begin{pmatrix} \sin 2t & 1+2\cos^2 t & 0 \\ -1-2\sin^2 t & -\sin 2t & 0 \\ -2\sin t & -2\cos t & 1 \end{pmatrix}$$

$X' = JX$: linearized system at x_1

3) $X_2 = \underbrace{(0,0,1)}_{p(t)} e^t$ $\quad$ $p(t)$ is a T-periodic solution

$\Rightarrow$ $\lambda_1 = e^{2\pi}$ is a characteristic multiplier.

Since, we can take $T = 2\pi$ and $\rho_1 = 1$

x_1 is a periodic solution $\Rightarrow$ $\lambda_2 = 1$.

We also have $\lambda_1 \lambda_2 \lambda_3 = e^{\int_0^T \mathrm{Trace}(J)\,ds}$

$\mathrm{Trace}(J) = 1$ $\Rightarrow$ $\lambda_1\lambda_2\lambda_3 = e^{2\pi}$ $\Rightarrow$ $\lambda_3 = 1$

4.) Apart from $\lambda_2 = 1$, we also have $\lambda_3 = 1$ and $\lambda_1 > 1$

$\Rightarrow$ The periodic solution is unstable

Problem 3:(25pts)
Consider the nonlinear system

$$\frac{dx}{dt} = -x - y + 2x(x^2+y^2),$$
$$\frac{dy}{dt} = x - y + y(x^2+y^2).$$

1)(8pts) Show that the system has no periodic solution inside the region
$R = \{(x,y)\in\mathbb{R}^2,\ 7x^2+5y^2 \le 1\}$.
2.)(12pts) Give all possible values of $a>0$ and $b>0$ such that the bounded set

$$D = \{(x,y)\in\mathbb{R}^2,\ a^2 \le x^2+y^2 \le b^2\}$$

is a trapping region of the system (that is, when a trajectory enters D it remains in D forever, or when a trajectory leaves D it will never return to D forever).
3.)(5pts) Deduce that the system has at least one closed orbit.

<u>Solution</u>

1) $\dot{x}=f(x,y)$, $\dot{y}=g(x,y)$ $\quad \nabla\cdot\begin{pmatrix}f\\g\end{pmatrix} = f_x+g_y = -2+7x^2+5y^2$
$= -1 \underbrace{-1+7x^2+5y^2}_{\le 0,\ \text{for all } (x,y)\in R}$

$\Rightarrow \nabla\cdot\begin{pmatrix}f\\g\end{pmatrix} \le -1,\ \forall (x,y)\in R$

There is no periodic solution in R by Bindixon's criteria

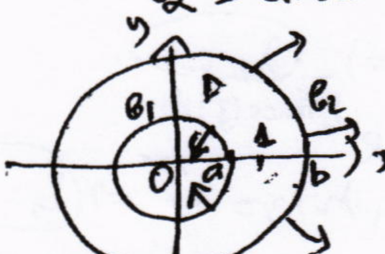
2.) $\begin{cases}\frac{1}{2}\frac{d}{dt}x^2 = -x^2-xy+2x^2(x^2+y^2)\\ \frac{1}{2}\frac{d}{dt}y^2 = xy-y^2+y^2(x^2+y^2)\end{cases} \Rightarrow \frac{1}{2}\frac{d}{dt}(x^2+y^2) = (2x^2+y^2-1)(x^2+y^2)$

We have, $2x^2+y^2-1 \ge x^2+y^2-1$. Let $b^2>1$ and $2a^2-1<0$

and $2x^2+y^2-1 \le 2(x^2+y^2)-1$.

C_1 = circle of center O and radius a, $\left.\frac{d(x^2+y^2)}{dt}\right|_{C_1} < 0$

C_2 = circle of centre O and radius b, $\left.\frac{d}{dt}(x^2+y^2)\right|_{C_2} > 0$



3) By Bindixon's theorem, the system has at least one periodic solution

Problem 4:(25pts) Let y, f and F be three scalar continuous functions on $\mathbb{R}$. Consider the first order differential equation

$$\frac{dy}{dt} + \frac{1}{t+1}y = F(y), \quad t \geq 0. \tag{1}$$

Assume that $|F(y)| \leq \gamma|y|$ and $|F(y_1) - F(y_2)| \leq \gamma|y_1 - y_2|$, for some $\gamma > 0$.
1.)(5pts) Multiplying Equation (1) by an integrating factor, show that

$$y(t) = \frac{y(0)}{t+1} + \int_0^t \frac{r+1}{t+1}F(y(r))dr, \ \forall t \geq 0.$$

2.)(10pts) Show that

$$|y(t)| \leq |y(0)|e^{\gamma t}, \ \forall t \geq 0.$$

3.) Consider two solutions y_1 and y_2 of Equation (1) such that $y_1(0) = y_2(0)$.
a.)(3pts) Write the differential equation satisfied by $v = y_1 - y_2$.
b.)(7pts) Given an arbitrary $T > 0$, show that

$$v(t) = 0, \ \forall t \in [0, T].$$

Solution

1) $\mu = e^{\int_0^t \frac{ds}{s+1}} = e^{\ln(t+1)} = t+1, \quad \forall t \geq 0$

We multiply the equation by $t+1$, to find

$$\frac{d}{dt}(y(t+1)) = (t+1)F(y) \Rightarrow y(t+1) = y_0 + \int_0^t (s+1)F(y(s))\,ds$$

$$\Rightarrow y(t) = \frac{y_0}{t+1} + \int_0^t \frac{s+1}{t+1}F(y(s))\,ds, \quad \forall t \geq 0$$

2) We have $\quad y(t)(t+1) = y_0 + \int_0^t (s+1)F(y(s))\,ds$

$$\Rightarrow |y(t)|(t+1) \leq |y_0| + \int_0^t (s+1)|F(y(s))|\,ds$$

$$\leq |y_0| + \gamma\int_0^t (s+1)|y(s)|\,ds$$

We apply the Gronwall's inequality, to find

$$|y(t)|(t+1) \leq |y_0|e^{\gamma t}, \ \forall t \geq 0$$

$$\Rightarrow |y(t)| \leq \frac{|y_0|}{t+1}e^{\gamma t} \leq |y_0|e^{\gamma t}, \quad \forall t \geq 0$$

3) a) $v = y_1 - y_2 \Rightarrow \frac{dv}{dt} + \frac{1}{t+1}v = F(y_1) - F(y_2)$

b) Proceeding like above, we find

$$|v(t)|(t+1) \leq |v(0)| + \gamma\int_0^t (s+1)|v(s)|\,ds$$

We apply the Gronwall's inequality to find

$$\Rightarrow (t+1)|v(t)| \leq |v(0)|e^{\gamma t}, \quad \forall t \in [0,T]$$

$$\Rightarrow |v(t)| = 0, \quad \forall t \in [0,T]$$

$$\Rightarrow v(t) = 0, \quad \forall t \in [0,T]$$

Problem 5:(25pts) Consider the nonlinear system

$$\frac{dx}{dt} = -x(1-x),$$
$$\frac{dy}{dt} = (x^2 + y^2 - \frac{1}{4})y.$$

1.)(5pts) Find all critical points of the system.
1.)(12pts) Use Lyapunov direct method to show that the origin is asymptotically stable.
2.)(8pts) Study the stability of the point $A(1,0)$.

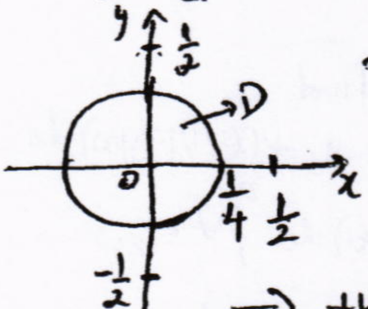
<u>Solution</u>

1.) $\begin{cases} x(1-x)=0 \\ (x^2+y^2-\frac{1}{4})y=0 \end{cases} \Longrightarrow \boxed{x=0}$ or $x=1$

$x=0 \Rightarrow y=-\frac{1}{2}, \frac{1}{2}$ or 0 $\qquad x=1 \Rightarrow (y^2+\frac{3}{4})y=0 \Rightarrow y=0$

We have critical points: 0, $A\begin{pmatrix}1\\0\end{pmatrix}$, $B\begin{pmatrix}0\\-\frac{1}{2}\end{pmatrix}$ and $C\begin{pmatrix}0\\\frac{1}{2}\end{pmatrix}$

2.) $\begin{cases} \frac{1}{2}\frac{d}{dt}x^2 = -x^2(1-x) \\ \frac{1}{2}\frac{d}{dt}y^2 = (x^2+y^2-\frac{1}{4})y^2 \end{cases} \Rightarrow \frac{1}{2}\frac{d}{dt}\underbrace{(x^2+y^2)}_{V} = \underbrace{-x^2(1-x)+y^2\left[x^2+y^2-\frac{1}{4}\right]}_{V^*}$



$D = \{(x,y)\in\mathbb{R}^2, x^2+y^2 \le \frac{1}{16}\}$

V is positive definite on D

V^* is negative definite on D

$\Longrightarrow$ the origin is

3.) $\underset{dt\,(x,y)}{\text{Jacobian}} = \begin{pmatrix} -1+2x & 0 \\ 2xy & x^2+3y^2-\frac{1}{4} \end{pmatrix}$, $J(1,0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$

the eigenvalues of J are $\lambda_1 = 1$, $\lambda_2 = 3/4$.

$\Rightarrow$ the point A is unstable.