King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
ODE Comprehensive Exam
The First Semester of 2021-2022 (211)
Time Allowed: 120mn

Name: ID number:

This is a closed book exam

Problem #	Marks	Maximum Marks
1		25
2		25
3		25
4		25
5		25
Total		$25 \times 4 = 100$

Solve only 4 problems of your choice.

Remark: In case a student solves all 5 problems, only the first 4 on the exam sheets will be graded.

Problem 1:(25pts)
1.)(12pts) Find the explicit solution of the IVP

$$\frac{dy}{dx} = y(y+1), \quad x, y \in \mathbb{R},$$
$$y(0) = 1,$$

and indicate the largest interval of definition of this solution.
2.)(13pts) Show that the IVP

$$\frac{dy}{dx} = -\frac{1}{(y - x^2 + 1)^2},$$
$$y(0) = 0,$$

has a unique solution in some interval around $x = 0$.

Solution

1.) $\displaystyle\int \frac{dy}{y(y+1)} = \int dx$, if $y \neq -1, 0$

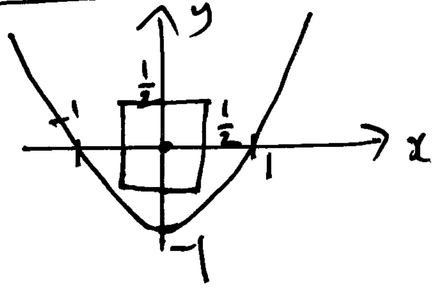
$\displaystyle\int \left(\frac{1}{y} - \frac{1}{y+1}\right) dy = x + C$

$\ln\left|\frac{y}{y+1}\right| = x + C \Rightarrow \frac{y}{y+1} = ce^x \Rightarrow y = \frac{ce^x}{1 - ce^x}$

Now, $y(0) = 1 \Rightarrow \frac{c}{1-c} = 1 \Rightarrow c = \frac{1}{2}$

Thus, $\boxed{y = \frac{e^x}{2 - e^x}, \; x \in (-\infty, \ln 2)}$

2.) $f(x,y) = \frac{1}{(y - x^2 + 1)^2}$



$D = \left\{(x,y), -\frac{1}{2} \le x \le \frac{1}{2}, -\frac{1}{2} \le y \le \frac{1}{2}\right\}$

$\frac{\partial f}{\partial y}(x,y) = \frac{-2}{(y - x^2 + 1)^3}$.

Now, if $(x,y) \in D$, then $0 \le x^2 \le \frac{1}{4} \Rightarrow -\frac{1}{4} \le -x^2 \le 0$
$\Rightarrow \frac{3}{4} \le 1 - x^2 \le 1 \Rightarrow \frac{1}{4} \le y - x^2 + 1 \le \frac{3}{2}$.

Thus, $|f(x,y)| \le 16$
and $\left|\frac{\partial f}{\partial y}(x,y)\right| \le 128$. Also, f and $\frac{\partial f}{\partial y}$ are continuous on D.
$\Rightarrow$ The IVP has a unique solution on $I = [-\frac{1}{32}, \frac{1}{32}$

Problem 2:(25pts) Consider the periodic system

$$\frac{dx}{dt} = -y(x^2+y^2)^2,$$
$$\frac{dy}{dt} = x(x^2+y^2)^2$$
$$\frac{dz}{dt} = 2z - 4(x^2+y^2).$$

1.)(10pts) Write the linearized system at the periodic solution $X_1(t) = (\cos t, \sin t, 2)$.
2.)(10pts) Find all the characteristics multipliers of the linearized system at $X_1(t)$.
3.)(5pts) Deduce the stability of the periodic solution $X_1(t)$.

Solution

1) $J = \begin{pmatrix} -4xy(x^2+y^2) & -(x^2+y^2)^2 - 4y^2(x^2+y^2) & 0 \\ (x^2+y^2)^2 + 4x^2(x^2+y^2) & 4xy(x^2+y^2) & 0 \\ -8x & -8y & 2 \end{pmatrix}$, $J = \begin{pmatrix} -2\sin 2t & -1-4\sin^2 t & 0 \\ 1+4\cos^2 t & 2\sin 2t & 0 \\ -8\cos t & -8\sin t & 2 \end{pmatrix}$

$X' = JX$ is the linearized system at X_1

2) We can see that $X_2(t) = \underbrace{(0,0,1)}_{P(t)} e^{2t}$ is another solution of the system.

$P(t)$ is a 2π-periodic $\Rightarrow$ $\lambda_1 = e^{4\pi}$ is a characteristic multiplier

Since X_1 is a periodic solution $\Rightarrow$ $\lambda_2 = 1$ is another multiplier

Last, we have the relation $\lambda_1\lambda_2\lambda_3 = e^{\int_0^{2\pi} \mathrm{Trace}(J)\, ds}$

$\mathrm{Trace}(J) = 2 \Rightarrow \lambda_1\lambda_2\lambda_3 = e^{4\pi} \Rightarrow \lambda_3 = 1$

3.) As $\lambda_1 > 1 \Rightarrow$ The periodic solution is unstable.

Problem 3:(25pts)
Consider the nonlinear system

$$\frac{dx}{dt} = 4x + y - 3x(x^2 + y^2),$$
$$\frac{dy}{dt} = -x + 4y - 4y(x^2 + y^2).$$

1)(9pts) Say whether it is possible or not to have a periodic solution of the system inside the region $R = \{(x, y) \in \mathbb{R}^2,\ 13x^2 + 15y^2 \leq 1\}$? Justify your answer.
2.) Let the function $V(x, y) = x^2 + y^2$ and consider the sets

$$C_1 = \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 = 1\}$$
$$C_2 = \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 = 4\}$$
$$D = \{(x, y) \in \mathbb{R}^2,\ 1 \leq x^2 + y^2 \leq 4\}.$$

a.)(4pts) Show that $\frac{d}{dt}V(x, y) = 2(x^2 + y^2)(4 - 3x^2 - 4y^2)$.
b.)(4pts) Show that $\frac{d}{dt}V(x, y) \leq 0$, for all $(x, y) \in C_2$.
c.)(4pts) Show that $\frac{d}{dt}V(x, y) \geq 0$ for all $(x, y) \in C_1$.
d.)(4pts) Admitting that D is a trapping region (also called Bendixson region), what conclusion can we draw from the Poincaré Bindixson theorem?

Solution

1.) $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$, $\nabla\cdot\begin{pmatrix} f \\ g \end{pmatrix} = f_x + g_y = 8 - 13x^2 - 15y^2$

$$= 7 + \underbrace{1 - 13x^2 - 15y^2}_{\geq 0,\ \text{for } (x,y) \in R}$$

$$\Rightarrow \nabla\cdot\begin{pmatrix} f \\ g \end{pmatrix} \geq 7,\ \forall (x,y) \in R.$$

There is no periodic solution in R by Bindixson Criteria

2.) a.) $\frac{1}{2}\frac{d}{dt}x^2 = 4x^2 + xy - 3x^2(x^2+y^2)$
$\frac{1}{2}\frac{d}{dt}y^2 = -xy + 4y^2 - 4y^2(x^2+y^2)$ $\Rightarrow \frac{1}{2}\frac{d}{dt}V = (x^2+y^2)(4 - 3x^2 - 4y^2)$

b.) $3(x^2+y^2) \leq 3x^2 + 4y^2 \Rightarrow 4 - 3x^2 - 4y^2 \leq 4 - 3(x^2+y^2)$

$$\Rightarrow \frac{dV}{dt} \leq 2(x^2+y^2)[4 - 3(x^2+y^2)] \Rightarrow \left.\frac{dV}{dt}\right|_{C_2} \leq 0$$

c.) $3x^2 + 4y^2 \leq 4(x^2+y^2) \Rightarrow 4 - 3x^2 - 4y^2 \geq 4 - 4(x^2+y^2)$

$$\Rightarrow \frac{dV}{dt} \geq 2(x^2+y^2)[4 - 4(x^2+y^2)] \Rightarrow \left.\frac{dV}{dt}\right|_{C_1} \geq 0$$

d.) The system has at least one periodic solution.

Problem 4:(25pts) Let y be a positive scalar continuous function on $[0, \infty)$. Consider the first order differential inequality

$$\frac{dy}{dt} + 2y^2 = 4y + 1, \qquad t \geq 0. \tag{1}$$

1.)(5pts) Using the Young inequality, deduce from Equation (1) that

$$\frac{dy}{dt} + y^2 \leq 5, \qquad \forall t \geq 0. \tag{2}$$

2.)(10pts) Multiplying (2) by an integrating factor, show that, if $\int_0^t y(s)ds \geq \alpha$, then

$$y(t) \leq y(0)e^{-\alpha} + 5t, \quad \forall t \geq 0.$$

3.) Consider two solutions y_1 and y_2 of Equation (1) such that $y_1(0) = y_2(0)$ and $y_1^2 - y_2^2 \geq 0$.
a.)(4pts) Write the differential equation satisfied by $v = y_1 - y_2$.
b.)(6pts) Given an arbitrary $T > 0$, show that

$$v(t) = 0, \quad \forall t \in [0, T].$$

Solution

1) Young inequality: $2|y| \leq \frac{y^2}{2} + 2 \Rightarrow 4|y| \leq y^2 + 4$

$4y \leq 4|y| \Rightarrow \frac{dy}{dt} + 2y^2 \leq y^2 + 4 + 1 \Rightarrow \frac{dy}{dt} + y^2 \leq 5.$

2.) We multiply ② by $e^{\int_0^t y(s)ds} \Rightarrow \frac{d}{dt}\left[y(t)e^{\int_0^t y(s)ds}\right] \leq 5e^{\int_0^t y(s)ds}$

Now, we integrate between 0 and t, to find

$y(t)e^{\int_0^t y(s)ds} - y(0) \leq 5\int_0^t e^{\int_0^r y(s)ds}dr$

$\Rightarrow y(t) \leq \left(y(0) + 5e^{\int_0^t y(s)ds}t\right)e^{-\int_0^t y(s)ds} \leq y(0)e^{-\int_0^t y(s)ds} + 5t$

$\leq y(0)e^{-\alpha} + 5t$

3.) $\left.\begin{array}{l}\frac{dy_1}{dt} + 2y_1^2 = 4y_1 + 1 \\ \frac{dy_2}{dt} + 2y_2^2 = 4y_2 + 1\end{array}\right\} \Rightarrow$ $y = y_1 - y_2$, $\frac{dy}{dt} + 2\underbrace{(y_1^2 - y_2^2)}_{\geq 0} = 4y$

$\Rightarrow \frac{dy}{dt} \leq 4y \Rightarrow \frac{d}{dt}\left[ye^{-4t}\right] \leq 0$

$\Rightarrow ye^{-4t} \leq y(0) \Rightarrow y(t) \leq y(0)e^{4t} \leq \underbrace{y(0)}_{=0}e^{4T}$

$\Rightarrow y(t) = 0, \forall t \in [0, T]$

Problem 5:(25pts) Consider the nonlinear system

$$\frac{dx}{dt} = x(x^2 + y^2 - 1),$$
$$\frac{dy}{dt} = -y(1 - y^2).$$

1.)(6pts) Find all the critical points of the system.
2.)(7pts) Study the stability of the point $C(1,0)$.
3.) Let the function $V(x, y) = x^2 + y^2$ and the disk $D = \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 \le \frac{1}{4}\}$.
a.)(4pts) Compute the derivative $\frac{d}{dt}V(x, y)$.
b.)(4pts) Show that $\frac{d}{dt}V(x, y) < 0$, for all $(x, y) \in D$.
c.)(4pts) Deduce the stability of the origin.

Solution

1.) $\begin{cases} x(x^2+y^2-1)=0 \Rightarrow x=0 \text{ or } x^2+y^2=1 \\ -y(1-y^2)=0 \Rightarrow y=0, \text{ or } y=\pm 1 \end{cases} \Rightarrow \boxed{O\begin{pmatrix}0\\0\end{pmatrix}, A\begin{pmatrix}0\\-1\end{pmatrix}, B\begin{pmatrix}0\\1\end{pmatrix}, C\begin{pmatrix}1\\0\end{pmatrix}, D\begin{pmatrix}-1\\0\end{pmatrix}}$

2.) $J=\begin{pmatrix} 3x^2+y^2-1 & 2xy \\ 0 & 3y^2-1 \end{pmatrix} \Rightarrow J(C)=\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$

$\begin{vmatrix} \lambda-2 & 0 \\ 0 & \lambda+1 \end{vmatrix} = 0 \quad (\lambda-2)(\lambda+1)=0, \quad \lambda=-1, \lambda=2$

The critical point C is unstable.

3.) a) $\frac{1}{2}\frac{d}{dt}x^2 = x^2(x^2+y^2-1)$
$\frac{1}{2}\frac{d}{dt}y^2 = -y^2(1-y^2)$ $\Rightarrow \frac{1}{2}\frac{d}{dt}V = -y^2(1-y^2) + x^2(x^2+y^2-1)$
$= -[y^2(1-y^2) + x^2(1-x^2-y^2)]$

b) $1-y^2 > 0$ and $1-x^2-y^2 = \frac{3}{4} + \underbrace{\frac{1}{4} - x^2 - y^2}_{\geq 0} \geq \frac{3}{4}$
for all $(x,y) \in D$.
$\Rightarrow \frac{dV}{dt} < 0$ on D.

c.) V is positive definite on D and $\frac{dV}{dt}$ is negative definite on D $\Rightarrow$ The origin is asymptotically stable