King Fahd University of Petroleum and Minerals
Department of Mathematics
ODE Comprehensive Exam
The Second Semester of 2022-2023 (222)
Time Allowed: 150mn

Name: ID number:

This is a closed book exam

Problem #	Marks	Maximum Marks
1		25
2		25
3		25
4		25
5		25
Total		$25 \times 4 = 100$

Solve only 4 problems of your choice.

Remark: **In case a student solves all 5 problems, only the first 4 on the exam sheets will be graded.**

Problem 1:

1.)(12pts) Consider the IVP

$$\frac{dy}{dx} = (2y-1)(y-1),$$
$$y(0) = \frac{1}{3}.$$

Find the explicit solution of this IVP and indicate the largest interval of definition of the solution.

2.)(13pts) Show that the IVP

$$\frac{dy}{dx} = \sqrt{1-y} + \sqrt{1-x^2},$$
$$y(0) = \frac{1}{2},$$

has a unique solution in some interval around $x = 0$, and give this interval.

Solution: .

1.) $\int \frac{dy}{(2y-1)(y-1)} = \int dx$, $\int \left(\frac{-2}{2y-1} + \frac{1}{y-1} \right) dy = x + C$

$\ln|y-1| - \ln|2y-1| = x + C$

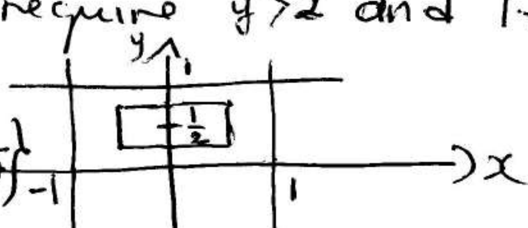
$\ln\left|\frac{y-1}{2y-1}\right| = x + C$, $\frac{y-1}{2y-1} = C e^{x}$

$y = \frac{1-Ce^{x}}{1-2Ce^{x}}$; $\frac{1}{3} = \frac{1-C}{1-2C}$, $C = 2$,

$$\boxed{y = \frac{1-2e^{x}}{1-4e^{x}}, \quad x \in (-\infty, -2\ln 2)}$$

2.) $f(x,y) = \sqrt{1-y} + \sqrt{1-x^2}$. We require $y > 2$ and $1 - x^2 \geq 0$

$D = \{(x,y),\ |x| \leq \frac{1}{2},\ |y - \frac{1}{2}| \leq \frac{1}{4}\}$



f is continuous on D.

$0 \leq x^2 \leq \frac{1}{4}$

$-\frac{1}{4} \leq -x^2 \leq 0$, $\frac{3}{4} \leq 1 - x^2 \leq 1$, $\frac{\sqrt{3}}{2} \leq \sqrt{1-x^2} \leq 1$

$-\frac{1}{4} \leq y - \frac{1}{2} \leq \frac{1}{4}$, $\frac{1}{4} \leq y \leq \frac{3}{4}$, $-\frac{3}{4} \leq -y \leq -\frac{1}{2}$, $\frac{1}{4} \leq 1 - y \leq \frac{1}{2}$

$\Rightarrow |f(x,y)| \leq 1 + \frac{\sqrt{2}}{2}$ $\qquad \frac{1}{2} \leq \sqrt{1-y} \leq \frac{\sqrt{2}}{2}$

$\frac{\partial f}{\partial y} = \frac{-1}{2\sqrt{1-y}} \Rightarrow \left|\frac{\partial f}{\partial y}\right| = \frac{1}{2\sqrt{1-y}} \leq 1$ on D

$\Rightarrow$ The IVP has a unique solution $y = y(x)$, $x \in \left[\frac{-1}{4+2\sqrt{2}}, \frac{1}{4+2\sqrt{2}}\right]$

Problem 2: Consider the nonlinear system

$$\frac{dx}{dt} = -2y(x^2+y^2),$$
$$\frac{dy}{dt} = 2x$$
$$\frac{dz}{dt} = 3z + 3(x^2+y^2).$$

1.)(5pts) Show that system has a periodic solution of the form $X_1(t) = (\alpha\cos\omega t, \alpha\sin\omega t, \beta)$, for some $\alpha, \omega > 0$ and $\beta \in \mathbb{R}$.
2.)(20pts) Analyze the stability of the periodic solution $X_1(t)$ by using the Floquet theory.

Solution:

1.) $-\alpha\omega\sin\omega t = -2\alpha\sin\omega t(\alpha^2) \Rightarrow \alpha\omega = 2\alpha^3 \Rightarrow \alpha^2 = 1, \ \alpha = 1$

$\alpha\omega\cos\omega t = 2\alpha\cos\omega t \Rightarrow \alpha\omega = 2\alpha \Rightarrow \omega = 2$

$0 = 3\beta + 3\alpha^2 \Rightarrow \beta = -1$

Thus, $X_1(t) = (\cos 2t, \sin 2t, -1)$

2.) The Jacobian of the system at (x_0, y_0, z_0) is

$$J = \begin{pmatrix} -4x_0y_0 & -2(x_0^2+3y_0^2) & 0 \\ 2 & 0 & 0 \\ 6x_0 & 6y_0 & 3 \end{pmatrix}, \quad \underbrace{J(X_1(t))}_{A(t)} = \begin{pmatrix} -2\sin 4t & -2(1+2\sin^2 2t) & 0 \\ 2 & 0 & 0 \\ 6\cos 2t & 6\sin 2t & 3 \end{pmatrix}$$

$\dot{Z} = A(t)Z$ is the linearized system at $X_1(t)$.

On characteristic multipliers is $\rho_1 = 1$, as $\dot{X}_1(t)$ is one solution.

We write the linearized system as

$$\begin{cases} \dot{z}_1 = -2\sin 4t\, z_1 - 2(1+2\sin^2 t)z_2 \\ \dot{z}_2 = 2z_1 \\ \dot{z}_3 = 6\cos 2t\, z_1 + 6\sin 2t\, z_2 + 3z_3 \end{cases}$$

We can see that $Y(t) = (0, 0, e^{3t}) = (0,0,1)e^{3t}$ is another solution of the linearized system

$\Rightarrow \rho_2 = e^{3\pi}$ is another multiplier ; $A(t+\pi) = A(t)$

We also have $\underbrace{\rho_1 \rho_2}_{e^{3\pi}} \cdot \rho_3 = e^{\int_0^\pi \mathrm{Tr}(A)\,ds} = e^{\int_0^\pi (3-2\sin 4s)\,ds} = e^{3\pi}$

$\Rightarrow \rho_3 = 1$

The periodic solution X_1 is unstable, since $\rho_2 > 1$

Problem 3: Consider the nonlinear system

$$\frac{dx}{dt} = x(x^2+y^2-3x-4) - y,$$
$$\frac{dy}{dt} = y(x^2+y^2-3x-4) + x.$$

1)(5pts) Find a region in the xy-plane, where it is not possible to have a periodic solution.
2.)(20pts) Find a trapping region, where we have the guarantee that there exists a periodic solution of the system.

Solution:

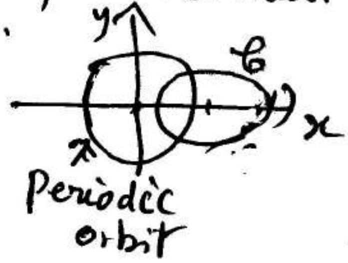
1) $f = x(x^2+y^2-3x-4)-y \Rightarrow f_x = 3x^2+y^2-6x-4$

$g = y(x^2+y^2-3x-4)+x \Rightarrow g_y = x^2+3y^2-3x-4$

Thus, $f_x+g_y = 4x^2+4y^2-9x-8 = 4\left[\left(x-\frac{9}{8}\right)^2-\frac{81}{64}\right]+4y^2-8$
$= 4\left(x-\frac{9}{8}\right)^2+4y^2-8-\frac{81}{16}$

From Bendixon's criteria, there is no periodic solution inside the ellipse $\mathcal{B} = \left\{4\left(x-\frac{9}{8}\right)^2+4y^2 < 8+\frac{81}{16}\right\}$

There is also no periodic solution outside $\mathcal{B}$. If there is a periodic orbit, then it must cross $\mathcal{B}$.



2.) $\frac{1}{2}\frac{d}{dt}x^2 = x^2(x^2+y^2-3x-4) - xy$

$+\quad \frac{1}{2}\frac{d}{dt}y^2 = y^2(x^2+y^2-3x-4) + xy$

$\frac{1}{2}\frac{d}{dt}(x^2+y^2) = (x^2+y^2)(x^2+y^2-3x-4)$

Let $x = r\cos\theta$, $y = r\sin\theta$ $\Rightarrow x^2+y^2 = r^2$

$\Rightarrow \underbrace{\frac{1}{2}\frac{d}{dt}r^2}_{r\frac{dr}{dt}} = r^2(r^2-3r\cos\theta-4)$

$\Rightarrow \frac{dr}{dt} = r\,(r^2-3r\cos\theta-4)$

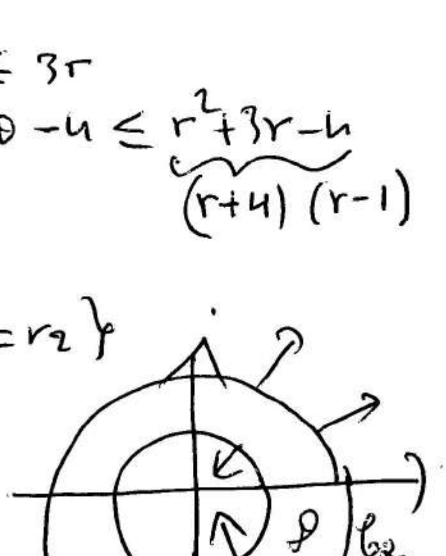
Now, $-1 \le \cos\theta \le 1 \Rightarrow -3r \le -3r\cos\theta \le 3r$

$\underbrace{r^2-3r-4}_{(r-4)(r+1)} \le r^2-3r\cos\theta-4 \le \underbrace{r^2+3r-4}_{(r+4)(r-1)}$

let $r_1 \in (0,1)$ and $\mathcal{C}_1 = \{r = r_1\}$

$r_2 \in (4,\infty)$ and $\mathcal{C}_2 = \{r = r_2\}$

$\left.\frac{dr}{dt}\right|_{\mathcal{C}_1} < 0$ and $\left.\frac{dr}{dt}\right|_{\mathcal{C}_2} > 0$



$\mathcal{Q} = \{r_1 \le r \le r_2\}$
is a Bendixon region

Problem 4: Consider the first order differential equation

$$\frac{dy}{dt} + y = -4y^3 + 1, \qquad t \geq 0. \tag{1}$$

1.)(5pts) Show that $y^2(t) \leq y^2(0) + 1$, $\forall t \geq 0$.
2.)(4pts) Show that $\int_0^t y^2(s)ds \leq y^2(0) + t$, $\forall t \geq 0$.
3.)(16pts) Show that Problem (1) cannot have more than two different solutions with the same initial condition $y(0)$.

Solution:

1.) $\frac{1}{2}\frac{dy^2}{dt} + y^2 = -4y^4 + y$; $|y| \leq \frac{1}{2}y^2 + \frac{1}{2}$ (Young inequality)

$\Rightarrow \frac{1}{2}\frac{dy^2}{dt} + \frac{1}{2}y^2 \leq \frac{1}{2} \Rightarrow \frac{dy^2}{dt} + y^2 \leq 1,$

$\Rightarrow y^2(t) - y^2(0) + \int_0^t y^2(s)\,ds \leq \int_0^t ds$, $\boxed{\int_0^t y^2(s)ds \leq y^2(0) + t, \ \forall t \geq 0}$

We also have

$\frac{d}{dt}(e^t y^2) \leq e^t$, $e^t y^2(t) - y^2(0) \leq \int_0^t e^s ds = e^t - 1$

$\Rightarrow y^2(t) \leq y^2(0)e^{-t} + 1 - e^{-t} \leq y^2(0) + 1$

$\boxed{y^2(t) \leq y^2(0) + 1, \ \forall t \geq 0}$

2.) Assume we have two solutions y_1 and y_2.
Let $y = y_1 - y_2$. So, $y(0) = y_1(0) - y_2(0) = 0$ (since $y_1(0) = y_2(0)$)
The function y satisfies the equation

$$\frac{dy}{dt} + y = -4\underbrace{(y_1^3 - y_2^3)}_{(y_1-y_2)(y_1^2+y_1y_2+y_2^2)}$$

$\Rightarrow$

$\frac{d}{dt}(ye^t) = -4y(y_1^2 + y_1y_2 + y_2^2)e^t$

$\Rightarrow ye^t - y(0) = -4\int_0^t y(y_1^2 + y_1y_2 + y_2^2)e^s ds$

$|y|e^t \leq |y(0)| + 4\int_0^t |y|(y_1^2 + |y_1y_2| + y_1^2)e^s ds$

$\leq |y(0)| + 6e^t\int_0^t |y|(y_1^2 + y_2^2)$; $|y_1y_2| \leq \frac{1}{2}(y_1^2 + y_2^2)$

$\Rightarrow |y(t)| \leq |y(0)| + 6\int_0^t |y(s)|(y_1^2 + y_2^2)\,ds$

We apply the Gronwall inequality:

$|y(t)| \leq |y(0)|e^{6\int_0^t (y_1^2+y_2^2)} \leq \overset{=0}{|y(0)|}e^{12y_1^2(0)+12t}$, $\forall t \in [0,T]$

$\Rightarrow y(t) = 0$, $\forall t \in [0,T]$, T arbitrary $\Rightarrow y(t) = 0$, $\forall t \geq 0$

Problem 5: Consider the nonlinear system

$$\frac{dx}{dt} = x(y^2 - 4),$$
$$\frac{dy}{dt} = -y(1 - x^2).$$

1.)(8pts) Find all the critical points of the system.
2.)(6pts) Study the stability of the critical point $C(1,2)$.
3.)(8pts) Find a closed domain of $\mathbb{R}^2$ containing no other critical point than the origin, and a positive definite function $V(x,y)$ such that $\frac{dV}{dt}(x,y)$ is negative definite on D.
4.)(3pts) Deduce the stability of the origin.

__Solution:__

1.) $\begin{cases} x(y^2-4)=0 \\ y(1-x^2)=0 \end{cases} \Rightarrow x=0$ or $y^2=4,\ y=\pm 2$

$\Downarrow \qquad\qquad \Downarrow$

$y=0 \qquad x^2=1,\ x=\pm 1$

The critical points are the origin, $A\begin{pmatrix}1\\2\end{pmatrix}, B\begin{pmatrix}-1\\2\end{pmatrix}, C\begin{pmatrix}1\\-2\end{pmatrix}, D\begin{pmatrix}-1\\-2\end{pmatrix}$

2.) The Jacobian of the system is

$$J(x_0,y_0) = \begin{pmatrix} y_0^2-4 & 2x_0y_0 \\ 2x_0y_0 & -(1-x_0^2) \end{pmatrix}, \quad J(A) = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$$

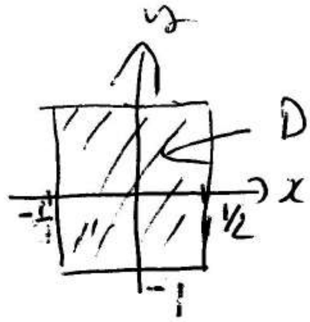
$$\begin{vmatrix} -\lambda & 4 \\ 4 & -\lambda \end{vmatrix} = \lambda^2 - 16 = 0 \Rightarrow \lambda = \pm 4 \Rightarrow A \text{ is unstable}$$

3.)
$$\frac{1}{2}\frac{d}{dt}x^2 = x^2(y^2-4)$$
$$+ \quad \frac{1}{2}\frac{d}{dt}y^2 = -y^2(1-x^2)$$

$$\frac{1}{2}\frac{d}{dt}(x^2+y^2) = x^2(y^2-4) - y^2(1-x^2)$$
$$= -x^2(4-y^2) - y^2(1-x^2)$$

$D\subseteq\{(x,y) /\ 4-y^2>0 \text{ and } 1-x^2>0\}$

For instance, $D=\{(x,y) /\ -\frac{1}{2}\le x\le\frac{1}{2},\ -1\le y\le 1\}$



4.) V is positive definite on $\mathbb{R}^2$
V^* is negative definite in D
$\Rightarrow$ the origin is asymptotically stable.