King Fahd University of Petroleum and Minerals
Department of Mathematics
ODE Comprehensive Exam
The First Semester of 2023-2024 (231)
Time Allowed: 150min

Name: ID number:

This is a closed book exam

Problem #	Marks	Maximum Marks
1		25
2		25
3		25
4		25
5		25
Total		$25 \times 4 = 100$

Solve only 4 problems of your choice.

Remark: In case a student solves all 5 problems, only the first 4 on the exam sheets will be graded.

Problem 1:

1.) Consider the IVP

$$(x^2-4)\sin^3 y\,dy = \frac{1}{\cos^2 y}dx,$$
$$y(0) = 0.$$

a.)(9pts) Find an implicit solution of the IVP. Write this solution without absolute value

b.)(4pts) Indicate the largest interval of definition of this solution.

2.) Consider the IVP

$$\frac{dy}{dx} = (x^2-2)y^{\frac{2}{3}} + 2,$$
$$y(1) = 0,$$

a.)(8pts) Show that the IVP has a solution in some interval to be given explicitely.

b.)(4pts) Is this solution unique? Justify your answer.

Solution: .

1.) $(x^2-4)\sin^3 y\, dy = \frac{1}{\cos^2 y}dx \Rightarrow \int \cos^2 y \sin^3 y\, dy = \int \frac{1}{x^2-4}dx$

Thus $\int \cos^2 y(1-\cos^2 y)\sin y\, dy = \frac{1}{4}\int\left(\frac{1}{x-2} - \frac{1}{x+2}\right)dx$

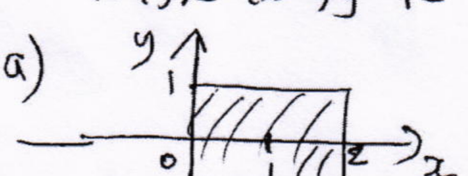
$-\frac{1}{3}\cos^3 y + \frac{1}{5}\cos^5 y = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$; $y(0)=0 \Rightarrow C = -\frac{2}{15}$

Thus $-\frac{1}{3}\cos^3 y + \frac{1}{5}\cos^5 y = \frac{1}{4}\ln\left(\frac{2-x}{x+2}\right) - \frac{2}{15}$, $x \in I \subset (-2,2)$

b)  The largest interval is $I =]\alpha, \beta[\subset (-2,2)$ $\boxed{\alpha, \beta \text{ are where } 6 \text{ meets } y = \frac{\pi}{2}, -\frac{\pi}{2}}$

2.) $f(x,y) = (x^2-2)y^{\frac{2}{3}} + 2$

• Let $D = \{(x,y),\ 0 \le x \le 2,\ -1 \le y \le 1\}$

a) 

f is continuous on D.

For any $(x,y) \in D$, $-2 \le x^2-2 \le 2$

$0 \le y^{2/3} \le 1$

$|f(x,y)| \le |x^2-2||y|^{\frac{2}{3}} + 2 \le 4.$

From the theorem of existence of a solution to an ODE, the IVP has a solution $y = y(x)$, $x \in [1-\alpha, 1+\alpha]$,

$\alpha = \min\{1, \frac{1}{4}\} = \frac{1}{4} \Rightarrow y = y(x)$, $x \in [\frac{3}{4}, \frac{5}{4}]$

b) $\frac{\partial f}{\partial y} = \frac{2}{3}(x^2-2)y^{-\frac{1}{3}}$ and $\frac{\partial f}{\partial y}$ is not continuous on D.

So, we can't decide on the uniqueness of the solution

Problem 2:

1.)(13pts) Find a periodic solution of the system

$$\frac{dx}{dt} = 2\pi y,$$
$$\frac{dy}{dt} = -2\pi x(x^2 + y^2),$$

and analyze its stability.

3.)(12pts) Find the Floquet multipliers of the system

$$X' = \begin{pmatrix} -1 & 1 \\ 0 & 1 + \cos t - \frac{\sin t}{2+\cos t} \end{pmatrix} X.$$

Solution:

1.) $\begin{cases} x = a\sin\omega t \\ y = a\cos\omega t \end{cases}$ $\quad \frac{dx}{dt} = a\omega\cos\omega t = 2\pi a\cos\omega t \Rightarrow a\omega = 2\pi \Rightarrow \omega = 2\pi$

$\frac{dy}{dt} = -a\omega\sin\omega t = -2\pi a^3\sin\omega t \Rightarrow a^2 = 1,\ a = \pm 1$

A periodic solution is $(x,y) = (\sin 2\pi t, \cos 2\pi t)$

The Jacobian of the system is $J = \begin{pmatrix} 0 & 2\pi \\ -2\pi(x^2+y^2+2x^2), & -4\pi xy \end{pmatrix}$

$A(t) = J(x(t), y(t)) = \begin{pmatrix} 0 & 2\pi \\ -2\pi(1+2\sin^2 2\pi t) & -4\pi\cos 2\pi t\sin 2\pi t \end{pmatrix}$

The linearized system $X' = A(t)X$. It has one characteristic multiplier $\rho_1 = 1$. Then, $\rho_1 \cdot \rho_2 = e^{-2\pi\int_0^1 \sin 4\pi s\,ds} = e^0 = 1 \Rightarrow \rho_2 = 1$

$\Rightarrow$ The periodic solution is stable

2.) The system is $\quad x' = -x + y,\ y' = \left(1 + \cos t - \frac{\sin t}{2+\cos t}\right) y$

$\int \frac{dy}{y} = \int \left(1 + \cos t - \frac{\sin t}{2+\cos t}\right) dt \Rightarrow \ln|y| = t + \sin t + \ln(2+\cos t) + c_1$

$\Rightarrow y = c_1(2+\cos t)e^{t+\sin t}$

Next, $\frac{d}{dt}(xe^t) = c_1(2+\cos t)e^{2t+\sin t} \Rightarrow xe^t = c_1\int(2+\cos t)e^{2t+\sin t}dt + c_2$

$= c_1 e^{2t+\sin t} + c_2$

$\Rightarrow x = c_1 e^{t+\sin t} + c_2 e^{-t}$

$\Rightarrow X = c_1 \underbrace{\begin{pmatrix} e^{\sin t} \\ (2+\cos t)e^{\sin t} \end{pmatrix}}_{P_1} e^t + c_2 \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{2} e^{-t} \Rightarrow$ The characteristic expone are $\lambda = 1$ and $\lambda = -1$

P_1 and P_2 are periodic of period $2\pi \Rightarrow \boxed{\rho_1 = e^{2\pi} \text{ and } \rho_2 = e^{-2\pi}}$ are characteristic multipliers.

Problem 3:
Consider the nonlinear system

$$\frac{dx}{dt} = -x(x^2 - xy + y^2) + 2x,$$
$$\frac{dy}{dt} = -3y(x^2 - xy + y^2) + 2(y - x).$$

1)(10pts) Consider the ellipse

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2,\ 3x^2 - 4xy + 5y^2 = 2\}.$$

If the system has a periodic solution, does the orbit of this periodic solution crosses $\mathcal{C}$ or no? Justify your response clearly.
2.)(15pts) Prove that the systems has a periodic solution.

Solution:

1.) $f = -x(x^2-xy+y^2)+2x \Rightarrow f_x = -(3x^2-2xy+y^2)+2$

$g = -3y(x^2-xy+y^2)+2(y-x) \Rightarrow g_y = -3(x^2-2xy+3y^2)+2$

$\nabla\cdot\binom{f}{g} = f_x+g_y = -6x^2+8xy-10y^2+4 = -2(3x^2-4xy+5y^2-2)$

Bendixon criteria $\Rightarrow$ There is no periodic solution inside $\mathcal{C}$ and exterior of $\mathcal{C}$. So, if the system has a periodic solution, its orbit must cross $\mathcal{C}$.

2.) $\frac{1}{2}\frac{dx^2}{dt} = -x^2(x^2-xy+y^2)+2x^2$

$\frac{1}{2}\frac{dy^2}{dt} = -3y^2(x^2-xy+y^2)+2y(y-x)$

$\frac{1}{2}\frac{d}{dt}(x^2+y^2) = -(x^2+3y^2)(x^2-xy+y^2)+2(x^2-xy+y^2)$

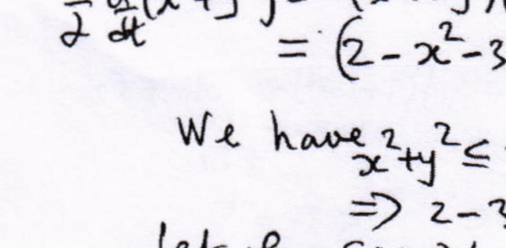
$= (2-x^2-3y^2)\underbrace{(x^2-xy+y^2)}_{\geq 0}$

We have $x^2+y^2 \leq x^2+3y^2 \leq 3(x^2+y^2)$

$\Rightarrow 2-3(x^2+y^2) \leq 2-x^2-3y^2 \leq 2-(x^2+y^2)$

let $\mathcal{C}_1 = \{(x,y)/\ x^2+y^2 = a^2\}$; $0 < 2-3a^2 \Rightarrow a < \sqrt{\frac{2}{3}}$

y $\mathcal{C}_2 = \{(x,y)/\ x^2+y^2 = b^2\}$; $2-b^2 < 0 \Rightarrow b > \sqrt{2}$



$D = \{(x,y)/\ a^2 < x^2+y^2 < b^2\}$ is a trapping region.

Poincaré Bendixon theorem $\Rightarrow$ The system has a periodic solution

Problem 4:

1.)(13pts) Consider the ODE

$$\frac{dy}{dx} + y = 25y^3 - 1, \quad x \in (-\infty, 0].$$

Show that

$$y^2(t) \leq y^2(0)e^{-t} - 1 + e^{-t}, \ \forall t \leq 0.$$

2.)(12pts) Consider the ODE

$$\frac{dy}{dx} + y^9 = 3y^3, \quad x \in [0, \infty).$$

Given that $\int_0^t y^2(x)dx \leq 3$, $\forall t \geq 0$, show that

$$y^2(t) \leq y^2(0)e^{18} \ \forall t \geq 0.$$

<u>Solution:</u>

1.) $\frac{1}{2}\frac{dy^2}{dx} + y^2 = \underbrace{25y^4}_{\geq 0} - y \Rightarrow \frac{1}{2}\frac{dy^2}{dx} + y^2 \geq -y$

Young inequality $\Rightarrow |y| \leq \frac{y^2}{2} + \frac{1}{2} \Rightarrow -y \geq -\frac{y^2}{2} - \frac{1}{2}$

Thus, $\frac{1}{2}\frac{dy^2}{dx} + y^2 \geq -\frac{y^2}{2} - \frac{1}{2}$; $\frac{dy^2}{dx} + y^2 \geq -1$

So, $\frac{d}{dx}[y^2 e^x] \geq -e^x \Rightarrow \int_t^0 \frac{d}{dx}(y^2 e^x)\,dx \geq -\int_t^0 e^x dx, \ \forall t \leq 0$

$\Rightarrow y^2(0) - y^2(t)e^t \geq e^t - 1 \Rightarrow \boxed{y^2(t) \leq y^2(0)e^{-t} + e^{-t} - 1, \ \forall t \leq 0}$

2.) $\frac{1}{2}\frac{dy^2}{dx} + \underbrace{y^{10}}_{\geq 0} = 3y^4 \Rightarrow \frac{1}{2}\frac{dy^2}{dx} \leq 3y^4$, $\frac{dy^2}{dx} \leq 6y^4$.

$\Rightarrow \frac{d}{dx}\left[y^2 e^{-6\int_0^x y^2(s)ds}\right] \leq 0 \Longrightarrow \int_0^t \frac{d}{dx}\left[y^2 e^{-6\int_0^x y^2 ds}\right]dx \leq 0$, $\forall t \geq 0$

$y^2(t)e^{-6\int_0^t y^2(s)ds} - y^2(0) \leq 0$

$\Rightarrow y^2(t) \leq y^2(0)e^{6\int_0^t y^2(s)ds}$

But, $6\int_0^t y^2(s)ds \leq 18 \Rightarrow e^{6\int_0^t y^2(s)ds} \leq e^{18}$

Thus, $\boxed{y^2(t) \leq y^2(0)e^{18}, \ \forall t \geq 0}$

Problem 5:

1.)(12pts) Consider the nonlinear system

$$\frac{dx}{dt} = \sqrt{x}(x^2 - y),$$
$$\frac{dy}{dt} = \frac{1}{y}(1 - x).$$

Study the stability of the critical point A(1.1).

2.)(13pts) Consider the nonlinear system

$$\frac{dx}{dt} = -x - 3y,$$
$$\frac{dy}{dt} = 3x.$$

Show that the origin is globally asymptotically stable.

Solution:

1.) The functions $f(x,y) = \sqrt{x}(x^2-y)$ and $g(x,y) = \frac{1}{y}(1-x)$ are differentiable in the neighborhood of the point A. Thus, we can decide on its stability by linearization

$J(1,1) = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$ is the Jacobian of the system at A.

$\begin{vmatrix} 2-\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(2-\lambda) - 1 = 0, \; \lambda_1 = 1-\sqrt{2}, \; \lambda_2 = 1+\sqrt{2}$

$\Rightarrow$ The critical point A is unstable.

2.)

$$\frac{1}{2}\frac{dx^2}{dt} = -x^2 - 3xy$$
$$\frac{1}{2}\frac{dy^2}{dt} = 3xy$$
$$\frac{1}{2}\frac{d}{dt}(x^2+y^2) = -x^2$$

$$y\frac{dx}{dt} = -xy - 3y^2$$
$$x\frac{dy}{dx} = 3x^2$$
$$\underbrace{y\frac{dx}{dt} + x\frac{dy}{dt}}_{\frac{d}{dt}(xy)} = 3x^2 - xy - 3y^2$$

$$\Rightarrow \frac{d}{dt}(x^2+y^2+\varepsilon xy) = -2x^2 + \varepsilon(3x^2 - xy - 3y^2) = (3\varepsilon-2)x^2 - \varepsilon xy - 3\varepsilon y^2$$

We now choose $\varepsilon > 0$ such that

$\varepsilon^2 - 4 < 0$ and $\varepsilon^2 + 4(3\varepsilon)(3\varepsilon - 2) < 0$

$\Rightarrow \varepsilon < \min\{2, \frac{24}{37}\}$. For instance $\varepsilon = 1/2$.

In this case, V is definite positive, and V^* is definite negative $\forall (x,y) \in \mathbb{R}^2 \Rightarrow 0$ is globally asymptotically stable.