King Fahd University of Petroleum and Minerals
Department of Mathematics
ODE Comprehensive Exam
The Second Semester of 2023-2024 (232)
Time Allowed: 180min

Name: ID number:

This is a closed book exam

Problem #	Marks	Maximum Marks
1		25
2		25
3		25
4		25
5		25
Total		$25 \times 4 = 100$

Solve only 4 problems of your choice.

Remark: In case a student solves all 5 problems, only the first 4 on the exam sheets will be graded.

Problem 1:

1.)Consider the IVP

$$\frac{dy}{dx} = y^2 - 4,$$
$$y(0) = 1.$$

a.)(9pts) Find an explicit solution of the IVP.

b.)(4pts) Indicate the largest interval of definition of this solution.

2.) Consider the IVP

$$\frac{dy}{dx} = \sqrt{1-x^2} + y^2,$$
$$y(0) = 0.$$

a.)(8pts) Show that the IVP has a solution in some interval I to be given explicitly.

b.)(4pts) Is this solution unique? Justify your answer.

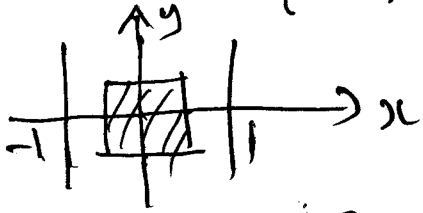
Solution: .

1.) $\frac{dy}{dx} = y^2-4$, $\int \frac{dy}{y^2-4} = \int dx$, $\frac{1}{4}\int\left(\frac{1}{y-2} - \frac{1}{y+2}\right)dy = x+C$

$\Rightarrow \frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| = x+C \Rightarrow \frac{y-2}{y+2} = ce^{4x}$, $y = \frac{2+2ce^{4x}}{1-ce^{4x}}$

$y(0)=1 \Rightarrow \frac{2+2c}{1-c} = 1$, $c = -\frac{1}{3}$, $\Rightarrow \boxed{y = \frac{2-\frac{2}{3}e^{4x}}{1+\frac{1}{3}e^{4x}},\ x\in(-\infty,\infty)}$

2.) a.) Consider $R = \{(x,y) \in \mathbb{R}^2;\ |x| \le \frac{1}{2},\ |y| \le \frac{1}{2}\}$



and $f(x,y) = \sqrt{1-x^2} + y^2$

• f is continuous on R

• $|x| \le \frac{1}{2} \Rightarrow 0 \le x^2 \le \frac{1}{4}$, $-\frac{1}{4} \le -x^2 \le 0$, $\frac{3}{4} \le 1-x^2 < 1$, $\frac{\sqrt{3}}{2} \le \sqrt{1-x^2} \le 1$

$|y| \le \frac{1}{2} \Rightarrow 0 \le y^2 \le \frac{1}{4} \Rightarrow |f(x,y)| \le 1 + \frac{1}{4} = \frac{5}{4}$

Thus, the IVP has a solution $y = y(x)$, $x \in]-\alpha, \alpha[$, where $\alpha = \min\left\{\frac{1}{2}, \frac{1/2}{5/4}\right\} = \frac{2}{5}$, $y = y(x), x \in]-\frac{2}{5}, \frac{2}{5}[$

b.) $\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial y}$ is continous in R

$\left|\frac{\partial f}{\partial y}\right| \le 2|y| \le 2\left(\frac{1}{2}\right) = 1 \Rightarrow f$ is Lipschitz on R

$\Rightarrow$ the IVP has a unique solution

Problem 2:

1.)(12pts) Analyze the stability of the periodic solution $X(t) = (\cos t, \sin t)$ of the system

$$\frac{dx}{dt} = -y(x^2+y^2),$$
$$\frac{dy}{dt} = x(x^2+y^2),$$

2.)(13pts) Find the characteristic multipliers of the system

$$X' = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1+\cos t & 0 \\ 1 & 1 & 0 \end{pmatrix} X,$$

given that the system has one periodic solution X_1 and another solution $X_2(t) = (2\sin t + \cos t, 1, 0)e^{2t}$.

Solution:

1.) The Jacobian ~~of the~~ system is $J(x,y) = \begin{pmatrix} -2xy & -x^2-3y^2 \\ 3x^2+y^2 & 2xy \end{pmatrix}$.

At the periodic solution

$X = (\cos t, \sin t) \Rightarrow A(t) = \begin{pmatrix} -\sin 2t & -1-2\sin^2 t \\ 1+2\cos^2 t & \sin 2t \end{pmatrix}$

$J(x(t), y(t)) = A(t)$

We now consider the system $x' = A(t)X$. (1)

System (1) has a periodic solution $\Rightarrow \rho_1 = 1$

If ρ_1, ρ_2 are the characteristic multipliers of (1), we have

$\rho_1 \cdot \rho_2 = e^{\int_0^{2\pi} \text{Trace}(A(t))\,dt} = e^0 = 1 \Rightarrow \rho_2 = 1$

$\Rightarrow$ The periodic solution is stable.

2.) the systeme $X' = A(t)X$ has a periodic solution

$\Rightarrow$ One characteristic multiplier $\boxed{\rho_1 = 1}$

The system has another solution

$X_2(t) = \underbrace{(2\sin t + \cos t, 1, 0)}_{\text{periodic}} e^{2t} \Rightarrow \boxed{\rho_2 = e^{4\pi}}$

Now, we have $\rho_1 \cdot \rho_2 \cdot \rho_3 = e^{\int_0^{2\pi} \text{Trace}(A(t))\,dt}$

$\text{Trace}(A(t)) = \cos t \Rightarrow \rho_1 \rho_2 \rho_3 = e^{\int_0^{2\pi} \cos t\,dt} = 1$

$\Rightarrow \rho_3 = \frac{1}{\rho_1 \cdot \rho_2} = \frac{1}{e^{4\pi}} \Rightarrow \boxed{\rho_3 = e^{-4\pi}}$

Problem 3:
Consider the nonlinear system

$$\frac{dx}{dt} = -6x + 3y + x(x^2 + y^2), \tag{1}$$
$$\frac{dy}{dt} = -3x - 6y + 2y(x^2 + y^2). \tag{2}$$

1)(10pts) Show that the system has no periodic solution inside the region

$$R = \{(x, y) \in \mathbb{R}^2,\ 5x^2 + 7y^2 < 12\}.$$

2.)(15pts) Prove that the systems (1)-(2) has at least one periodic solution.

Solution:

1.) let $f(x,y) = -6x + 3y + x(x^2+y^2)$ & $g(x,y) = -3x - 6y + 2y(x^2+y^2)$

$\Rightarrow f_x = -6 + 3x^2 + y^2$, $g_y = -6 + 2x^2 + 6y^2$

$\nabla\cdot\binom{f}{g} = f_x + g_y = -12 + 5x^2 + 7y^2$, and $\nabla\cdot\binom{f}{g} < 0$, $\forall (x,y) \in R$

From Bendixson criteria, there is no periodic solution inside R.

2.) $\frac{1}{2}\frac{d}{dt}x^2 = -6x^2 + 3xy + x^2(x^2+y^2)$

$\frac{1}{2}\frac{d}{dt}y^2 = -3xy - 6y^2 + 2y^2(x^2+y^2)$

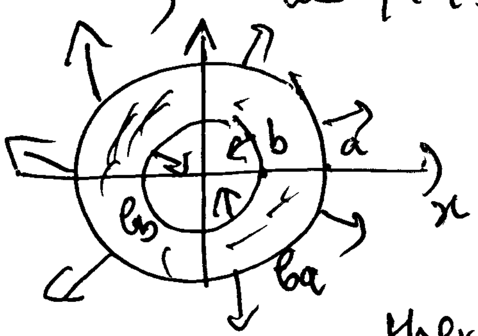
$\Rightarrow \frac{1}{2}\frac{d}{dt}(x^2+y^2) = -6(x^2+y^2) + (x^2+y^2)(x^2+2y^2)$
$= (-6 + x^2 + 2y^2)(x^2+y^2)$

We have $x^2+y^2 \le x^2+2y^2 \le 2(x^2+y^2)$

$\Rightarrow -6 + (x^2+y^2) \le -6 + x^2 + 2y^2 \le -6 + 2(x^2+y^2)$

let $C_1 = \{(x,y) \in \mathbb{R}^2 / x^2+y^2 = a^2,\ -6 + a^2 > 0\}$; $a > \sqrt{6}$

$C_2 = \{(x,y) \in \mathbb{R}^2 / x^2+y^2 = b^2,\ -6 + 2b^2 < 0\}$; $b < \sqrt{3}$



$D = \{(x,y) \in \mathbb{R}^2 / b^2 \le x^2+y^2 \le a^2\}$ is a trapping region.

From Poincaré–Bendixson theorem, there exists a periodic solution inside D.

Problem 4:

1.)(12pts) Consider the ODE

$$\frac{dy}{dt} + y = -y^5 + 2, \quad t \in [0, \infty).$$

Show that

$$y^2(t) \le y^2(0)e^{-t} + 4, \ \forall t \ge 0.$$

2.)(13pts) We assume that the following IVP

$$\frac{dy}{dt} = f(y) + \cos t,$$
$$y(0) = y_0.$$

has a solution $y = y(t)$, $\forall t \in [0, 2]$.
We also assume that the function f satisfies the estimate

$$|f(y_1) - f(y_2)| \le |y_1 - y_2|.$$

Show that this IVP has a unique solution.

Solution:

1.) We multiply the equation by y and we find

$$\frac{1}{2}\frac{d}{dt}y^2 + y^2 = -y^6 + 2y \Rightarrow \frac{1}{2}\frac{d}{dt}y^2 + y^2 \le 2y \le \underbrace{\frac{1}{2}y^2 + 2}_{\text{Young's inequality}}$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}y^2 + \frac{1}{2}y^2 \le 2 \Rightarrow \frac{d}{dt}y^2 + y^2 \le 4 \quad (1)$$

We multiply (1) by $e^t \Rightarrow \frac{d}{dt}(y^2e^t) \le 4e^t \Rightarrow$

$$\Rightarrow y^2(t)e^t - y^2(0) \le \underbrace{4\int_0^t e^s ds}_{4(e^t-1)} \Rightarrow y^2(t) \le y^2(0)e^{-t} + 4 - 4e^{-t}$$

$$\boxed{y^2(t) \le y^2(0)e^{-t} + 4, \ \forall t \ge 0}$$

2) Let y_1 and y_2 be two solutions of the IVP.

Let $y = y_1 - y_2$. $\Rightarrow \frac{dy}{dt} = f(y_1) - f(y_2)$ and $y(0) = 0$

We multiply this equation by $y \Rightarrow \frac{1}{2}\frac{d}{dt}y^2 = (f(y_1) - f(y_2))y$

But $|(f(y_1) - f(y_2))y| \le |y|^2 \Rightarrow \frac{1}{2}\frac{d}{dt}y^2 \le y^2$

$$\Rightarrow \frac{d}{dt}y^2 \le 2y^2 \Rightarrow \frac{d}{dt}(y^2e^{-2t}) \le 0 \Rightarrow y^2(t)e^{-2t} - y^2(0) \le 0$$

$$\Rightarrow y^2(t) \le \underbrace{y^2(0)}_{=0}e^{2t} \le y^2(0)e^4, \ t \in [0,2] \Rightarrow y^2(t) = 0, \ \forall t \in [0, 2]$$

$$\Rightarrow \boxed{y(t) = 0, \ t \in [0, 2]}$$

Problem 5:

1.) Consider the nonlinear system

$$\frac{dx}{dt} = x(4x^2 - y^2),$$
$$\frac{dy}{dt} = y(1 - x).$$

a.)(4pts) Find all the critical points of the system.
b.)(8pts) Study the stability of the point $A(1, 2)$.
2.) Consider the nonlinear system

$$\frac{dx}{dt} = 3y,$$
$$\frac{dy}{dt} = -2y - 3x.$$

a.)(9pts) Let $V(x, y) = x^2 + xy + y^2$. Compute $\frac{d}{dt}V(x(t), y(t))$.
b.)(4pts) Deduce the stability of the origin from part a.).

Solution:

1.) a.) Critical points satisfy $\begin{cases} x(4x^2-y^2)=0 \Rightarrow x=0 \text{ or } y=\pm 2x \\ y(1-x)=0 \Rightarrow y=0 \text{ or } x=1 \end{cases}$

$\Rightarrow \boxed{O(0,0),\ A(1,2)\ \ B(1,-2)}$

b) The Jacobian at the point (x_0, y_0) is $J = \begin{pmatrix} 12x_0^2 - y_0^2 & -2x_0y_0 \\ -y_0 & 1-x_0 \end{pmatrix}$

At the point A, $J = \begin{pmatrix} 8 & -4 \\ -2 & 0 \end{pmatrix}$. $\begin{vmatrix} 8-\lambda & -4 \\ -2 & -\lambda \end{vmatrix} = 0$, $-\lambda(8-\lambda) - 8 = 0$

$\lambda^2 - 8\lambda - 8 = 0$, $\lambda_1 = \frac{8-\sqrt{96}}{2} < 0$, $\lambda_2 = \frac{8+\sqrt{96}}{2} > 0$

the point A is unstable

2.) $\frac{1}{2}\frac{dx^2}{dt} = 3xy \Rightarrow \frac{dx^2}{dt} = 6xy$ (1)

$\frac{1}{2}\frac{dy^2}{dt} = -2y^2 - 3xy \Rightarrow \frac{dy^2}{dt} = -4y^2 - 6xy$ (2)

But, $\frac{d}{dt}xy = x\frac{dy}{dt} + y\frac{dx}{dt} = x(-2y-3x) + y(3y)$
$= -2xy - 3x^2 + 3y^2$ (3)

(1) + (2) + (3) $\Rightarrow \frac{d}{dt}V = -y^2 - 2xy - 3x^2 = -(\underbrace{y^2+2xy+3x^2}_{>0}) = V^*$

V is positive definite on $\mathbb{R}^2$
V^* is negative definite on $\mathbb{R}^2$
$\Rightarrow$ the origin is globally asymptotically stable.