Solution of
Comprehensive Exam (T232)

1) Compute ${}^{c}D_0^\alpha f$, $0 < \alpha < 1$, $t > 0$, for

$$f(t) = \begin{cases} t, & t < 1, \\ 1 - t, & t \geq 1. \end{cases}$$

Solution

For $t < 1$,

$${}^{c}D_0^\alpha t = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}.$$

For $t \geq 1$,

Since f is not continuous at $t = 1$, we use the definition

$${}^{c}D_0^\alpha f := {}^{RL}D_0^\alpha [f - f(0)] = {}^{RL}D_0^\alpha f = D I_0^{1-\alpha} f.$$

$$\Gamma(1-\alpha) I_0^{1-\alpha} f = \int_0^t (t-s)^{-\alpha} f(s)\, ds = \int_0^1 (t-s)^{-\alpha} f(s)\, ds + \int_1^t (t-s)^{-\alpha} f(s)\, ds$$
$$= \int_0^1 (t-s)^{-\alpha} s\, ds + \int_1^t (t-s)^{-\alpha}(1-s) ds$$

Using integration by parts,

$$\int_0^1 (t-s)^{-\alpha} s\, ds = \frac{(t-1)^{1-\alpha}}{\alpha-1} - \frac{1}{\alpha-1}\int_0^1 (t-s)^{1-\alpha}\, ds$$
$$= \frac{(t-1)^{1-\alpha}}{\alpha-1} - \frac{1}{\alpha-1}\left[\frac{(t-s)^{2-\alpha}}{\alpha-2}\right]_0^1$$
$$= \frac{(t-1)^{1-\alpha}}{\alpha-1} - \frac{1}{(\alpha-1)(\alpha-2)}[(t-1)^{2-\alpha} - t^{2-\alpha}]$$
$$\int_1^t (t-s)^{-\alpha} f(s)\, ds = I_1^{1-\alpha}(1-s) = \frac{(1-t)^{2-\alpha}}{\Gamma(2-\alpha)}$$

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2) Let $f \in L^1(a,b)$. Consider the Riemann-integral of order $\alpha > 0$,

$$(I_{b-}^\alpha f)(x) := \frac{1}{\Gamma(\alpha)}\int_x^b (s-x)^{\alpha-1} f(s)\, ds, \qquad x < b.$$

Show that $I_{b-}^\alpha f \in AC[a,b]$ for $\alpha > 1$.

Solution

Since $f \in L^1(a,b)$, then $I_{b-}^{\alpha-1} f \in L^1(a,b)$. Thus

$$I_{b-}^\alpha f = I_{b-} I_{b-}^{\alpha-1} f \in AC[a,b].$$

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3) Let $f \in C[a,\infty)$ and $1 < \alpha < 2$. Show that $u \in AC^2[0,\infty)$ is a solution of

$$ {}^{c}D_0^{\alpha} u(t) = f(t), \qquad t > 0, $$
$$ u(0) = u_0, \qquad u'(0) = u_1, $$

if and only if u is a solution of the problem

$$ u'(t) = I_0^{\alpha-1} f(t) + u_1, $$
$$ u(0) = u_0. $$

Solution

$\Longrightarrow$ Apply $I_0^{\alpha-1}$ to the differential equation. Then

$$ I_0^{\alpha-1}\, {}^{c}D_0^{\alpha} u = I_0^{\alpha-1} I_0^{2-\alpha} D^2 u = I_0 D^2 u = I_0 D u' = u' - u_1. $$

$\Longleftarrow$ Apply ${}^{C}D_0^{\alpha-1}$ to the integral equation,

$$ {}^{C}D_0^{\alpha-1} u'(t) = I_0^{2-\alpha} D u' = {}^{C}D_0^{\alpha} u. $$

Since $I_0^{\alpha-1} f(0) = 0$,

$$ {}^{C}D_0^{\alpha-1} I_0^{\alpha-1} f = {}^{RL}D_0^{\alpha-1} I_0^{\alpha-1} f = f. $$

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4) Solve the Cauchy problem

$$ {}^{c}D_0^{8/3} y(t) - 4y(t) = 0, \qquad t > 0, $$
$$ y(0) = 1, \qquad y'(0) = 0, \qquad y''(0) = 2. $$

Solution

Apply Laplace transform with $\alpha = 8/3$, $n = 3$, and let $Y(s) = \mathcal{L}\{y\}$,

$$ s^{\alpha} Y - s^{\alpha-1} y(0) - s^{\alpha-2} y'(0) - s^{\alpha-3} y''(0) - 4Y = 0. $$

$$ \Longrightarrow \quad Y = \frac{s^{\alpha-1}}{s^{\alpha}-4} + \frac{2s^{\alpha-3}}{s^{\alpha}-4}. $$

Applying inverse Laplace transform,

$$ y(t) = E_{\alpha}(4t^{\alpha}) + 2t^2 E_{\alpha,3}(4t^{\alpha}). $$

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5) Prove the following identity **or** show it is not correct,

$$ {}^{RL}D_0^{1/2}\, {}^{RL}D_0^{3/2} E_\alpha(x) = D^2 E_\alpha(x). $$

Solution

This identity is incorrect since

$$ D^2 x^k = k(k-1)x^{k-2}, \qquad k = 2, 3, \dots $$

$$ {}^{RL}D_0^{\frac{1}{2}}\, {}^{RL}D_0^{\frac{3}{2}}\, x^k = {}^{RL}D_0^{\frac{1}{2}} \frac{\Gamma(k+1)x^{k-\frac{3}{2}}}{\Gamma\left(k-\frac{1}{2}\right)}, \qquad k = 0, 1, \dots $$

$$ = \frac{k!\,\Gamma\left(k-\frac{1}{2}\right) x^{k-2}}{\Gamma\left(k-\frac{1}{2}\right)\,\Gamma(k-1)}, \qquad k = 0, 1, 2, \dots $$

$$ = \frac{k!\,x^{k-2}}{(k-2)!} = k(k-1)x^{k-2}, \qquad k = 2, 3, \dots $$

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6) Calculate the following derivative,

$$ {}^{c}D_0^{\alpha}(1+x)^{\alpha}, \qquad 0 \le x < 1, \qquad 0 < \alpha < 1. $$

Solution

By direct application of the definition or using the binomial theorem

Hint. $(x+y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k$, $|x| > |y|$.

$$ {}^{c}D_0^{\alpha}(1+x)^{\alpha} = {}^{c}D_0^{\alpha} \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = \sum_{k=1}^{\infty} (-1)^k \binom{\alpha}{k}\ {}^{c}D_0^{\alpha} x^k $$

$$ {}^{c}D_0^{\alpha} x^k = \frac{k!\,x^{k-\alpha}}{\Gamma(k-\alpha+1)}. $$

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7) Use successive approximation to compute $u_1(t)$ and $u_2(t)$ only for

$$ {}^{c}D_0^{\alpha} u = t + u^2, \qquad 0 < \alpha < 1, \qquad t > 0, $$

$$ u(0) = 0. $$

Solution Apply I_0^{α} to the equation,

$$ u(t) = \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} + I_0^{\alpha} u^2 = u_0(t) + I_0^{\alpha} u^2. $$

$$ u_1(t) = u_0(t) + I_0^{\alpha} u_0^2 = u_0(t) + \frac{1}{\Gamma^2(2+\alpha)} I_0^{\alpha} t^{2+2\alpha} = u_0(t) + \frac{\Gamma(3+2\alpha)\, t^{2+3\alpha}}{\Gamma^2(2+\alpha)\Gamma(3+3\alpha)}. $$

$$ u_2(t) = u_0(t) + I_0^{\alpha} u_1 = 2u_0(t) + I_0^{\alpha}\left[u_0(t) + \frac{\Gamma(3+2\alpha)\, t^{2+3\alpha}}{\Gamma^2(2+\alpha)\Gamma(3+3\alpha)}\right]. $$

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Formula

$$I_0^{\alpha}\,{}^{RL}D_0^{\alpha} f(t) = f(t) - \sum_{k=1}^{n} \frac{D_0^{\alpha-k} f(0)}{\Gamma(\alpha-k+1)}\ t^{\alpha-k}$$

$${}^{c}D_a^{\alpha} f \coloneqq {}^{RL}D_a^{\alpha}\left[f - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}\ (x-a)^k\ \right]$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}$$

$$\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\} = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}, \qquad \mathcal{L}\{1\} = 1/s$$

$$\mathcal{L}\{I_0^{\alpha} f(t)\} = s^{-\alpha}F(s)$$

$$\mathcal{L}\{{}^{RL}D_0^{\alpha} f\} = s^{\alpha}F(s) - \sum_{k=1}^{n} s^{k-1}\left(D_0^{\alpha-k} f\right)(0)$$

$$\mathcal{L}\{\,{}^{C}D_0^{\alpha} f\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}\left(D^{k} f\right)(0)$$
