King Fahd University of Petroleum and Minerals
Department of Mathematics
PDE Comprehensive Exam
The Second Semester of 2021-2022 (212)
Time Allowed: 150mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		25
2		25
3		25
4		25
5		25
Total		$25 \times 4 = 100$

Solve 4 problems of your choice.

Remark: **Only 4 problems will be taking into account, even if you solve the five problems.**

Problem 1:(25pts)

1.)(13pts) Use the method of characteristics to solve the Cauchy problem:

$$u_x + (1+y^2)u_y = u, \quad x, y \in \mathbb{R}, \tag{1}$$
$$u(0,y) = \tan^{-1} y. \tag{2}$$

2.)(12pts) Find the canonical form of the the parabolic PDE

$$u_{xx} - 2u_{xy} + u_{yy} = 0 \tag{3}$$

Solution:

1.) $\frac{dx}{dt} = 1 \quad \Rightarrow x = t + c_1$

$\frac{dy}{dt} = 1+y^2 \Rightarrow \int \frac{dy}{1+y^2} = t + c_2 \Rightarrow \tan^{-1} y = t + c_2$

$\frac{du}{dt} = u \quad \Rightarrow \int \frac{du}{u} = t + c_3 \Rightarrow \ln|u| = t + c$, or $u = c_3 e^t$

At $t=0$, $x=0$, $y=s$ and $u = \tan^{-1} s$.

thus, $c_1 = 0$, $c_2 = \tan^{-1} s$, $c_3 = \tan^{-1} s$.

This gives $x = t$, $\tan^{-1} y = t + \tan^{-1} s$, $u = e^t \tan^{-1} s$

$\Rightarrow \boxed{u = e^x(\tan^{-1} y - x)}$

2.) $\frac{dy}{dx} = -1 \quad \Rightarrow y + x = c_1$. Let $\xi = y + x$ and $\eta = x$

$\Rightarrow$ Jacobian of ξ, η $= \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$

Let $u(x,y) = w(\xi, \eta)$.

Thus, $u_x = w_\xi \xi_x + w_\eta \eta_x = w_\xi + w_\eta$

$u_y = w_\xi \xi_y + w_\eta \eta_y = w_\xi$

$u_{xx} = (w_\xi + w_\eta)_\xi \xi_x + (w_\xi + w_\eta)_\eta \eta_x = w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}$

$u_{xy} = u_{yx} = (w_\xi)_\xi \xi_x + (w_\xi)_\eta \eta_x = w_{\xi\xi} + w_{\xi\eta}$

$u_{yy} = (w_\xi)_\xi \xi_y + (w_\xi)_\eta \eta_y = w_{\xi\xi}$

$\Rightarrow \boxed{w_{\eta\eta} = w}$

Problem 2:(25pts)
1.)(12pts) Solve the nonhomogeneous problem

$$u_{tt} - 4u_{xx} = x \quad x \in \mathbb{R},\ t > 0, \tag{4}$$
$$u(x,0) = 0, \quad x \in \mathbb{R}, \tag{5}$$
$$u_t(x,0) = 0, \quad x \in \mathbb{R}. \tag{6}$$

Hint: You may write the problem satisfied by $v = u + \frac{x^3}{24}$ and apply the d'Alembert formula, or you may directly use the formula $u(x,t) = \frac{1}{4}\iint_\Delta X\,dA$, where Δ is the characteristic triangle at (x,t).
2.)(13pts) Solve the Cauchy problem

$$U_{tt} = U_{xx} + U_{yy} + U_{zz}, \quad (x,y,z) \in \mathbb{R}^3,\ t > 0, \tag{7}$$
$$u(x,y,z,0) = 0, \quad (x,y,z) \in \mathbb{R}^3, \tag{8}$$
$$u_t(x,y,z,0) = x, \quad (x,y,z) \in \mathbb{R}^3. \tag{9}$$

Hint: You can use the Kirchhoff formula $u(x,y,z,t) = \frac{1}{4\pi t}\iint_{S_t} u_t(X,Y,Z,0)\,d\sigma_t$, where S_t is the sphere of center (x,y,z) and radius t; and σ_t is the surface area of S_t.

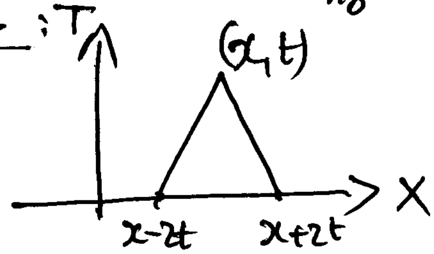
Solution:

1.) Method1: $u = v - \frac{x^3}{24}$, $v_{tt} - 4\left(v_{xx} - \frac{x}{4}\right) = x \Rightarrow v_{tt} - 4v_{xx} = 0$

Also, $v(x,0) = \frac{x^3}{24}$, $v_t(x,0) = 0 \xrightarrow[\text{formula}]{\text{d'Alembert}} v(x,t) = \frac{1}{48}\left[(x-2t)^3 + (x+2t)^3\right]$

Thus, $u(x,t) = \frac{1}{48}\left[(x-2t)^3 + (x+2t)^3\right] - \frac{x^3}{24}$

Method2:



$$u(x,t) = \frac{1}{4}\iint_\Delta X\,dA = \frac{1}{4}\int_0^t \int_{x-2(t-T)}^{x+2(t-T)} X\,dX\,dT$$
$$= x\int_0^t (t-T)\,dT = \frac{xt^2}{2}.$$

2.) $$u(x,y,z,t) = \frac{1}{4\pi t}\iint_{S_t} X\,d\sigma_t.$$

Let $\begin{cases} X = x + t\sin\theta\cos\varphi \\ Y = y + t\sin\theta\sin\varphi \\ Z = z + t\cos\theta \end{cases} \Rightarrow d\sigma_t = t^2\sin\theta\,d\theta\,d\varphi$, $\theta \in [0,\pi]$, $\varphi \in [0,2\pi]$

thus, $$u(x,y,z,t) = \frac{1}{4\pi t}\int_0^{2\pi}\int_0^{\pi}(x + t\sin\theta\cos\varphi)\,t^2\sin\theta\,d\theta\,d\varphi$$
$$= \frac{1}{4\pi t}\left[t^2 x\underbrace{\left(\int_0^\pi \sin\theta\,d\theta\right)}_{=2}2\pi + t^3\left(\int_0^\pi \sin^2\theta\,d\theta\right)\underbrace{\left(\int_0^{2\pi}\cos\varphi\,d\varphi\right)}_{=0}\right]$$

$$\Rightarrow u(x,y,z,t) = \frac{xt}{2\pi}.$$

Problem 3:(25pts)
1.)(15pts)Solve the initial value problem

$$u_t - u_{xx} = 0, \quad -\infty < x < \infty, \ t > 0. \tag{10}$$

$$u(x,0) = \begin{cases} 1, & \text{if } \ x \in [-1,1], \\ 0, & \text{elsewhere.} \end{cases} \tag{11}$$

Hint: Look for bounded solutions.
2.)(10pts) Consider the initial value problem

$$u_t - u_{xx} = 0, \quad 0 < x < 1, \ t > 0, \tag{12}$$

$$u(x,0) = 2, \qquad 0 \leq x \leq 1, \tag{13}$$

$$u(0,t) = u(1,t) = 2, \quad t \geq 0. \tag{14}$$

By using the weak maximum principle, what conclusion can we reach for the solution $u(x,t)$ of this problem? Justify your answer.

Solution:

1) Let $u = TX$. Thus, $T'X - TX'' = 0 \Rightarrow \frac{T'}{T} = \frac{X''}{X} = \lambda$.

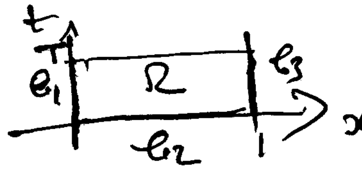
- If $\lambda = 0$, $X'' = 0$, $X = C_1 x + C_2$ X bounded $C_1 = 0 \Rightarrow u = C$
  $T' = 0$, $T = C$
- If $\lambda = w^2$, $X'' - w^2 X = 0$, $X = C_1 e^{wx} + C_2 e^{-wx}$, X bounded $\Rightarrow C_1 = C_2 = 0$
- If $\lambda = -w^2$, $X'' + w^2 X = 0$ $X = C_1 \cos wx \ \ C_2 \sin wx$
  $T' + w^2 T = 0$, $T = Ce^{-w^2 t}$

$\Rightarrow u(x,t) = \int_0^\infty (a_w \cos wx + b_w \sin wx)\, e^{-w^2 t}\, dw$

$\Rightarrow u(x,0) = \int_0^\infty (a_w \cos wx + b_w \sin wx)\, dw \Rightarrow \begin{cases} a_w = \frac{1}{\pi}\int_{-\infty}^{\infty} u(x,0) \cos wx\, dx = \frac{2 \sin w}{w} \\ b_w = \frac{1}{\pi}\int_{-\infty}^{\infty} u(x,0) \sin wx\, dx = 0 \end{cases}$

2.) Let $v = u - 2$.

Thus, $\begin{cases} v_t - v_{xx} = 0, & 0 < x < 1 \\ v(x,0) = 0, & 0 \leq x \leq 1 \\ v(0,t) = v(1,t) = 0, & t \geq 0 \end{cases}$. Let $\Omega = [0,1] \times [0,T)$, $T > 0$



- $v(x,t) = 0$ on $e_1 \cup e_2 \cup e_3$
- v is continuous on Ω

By the weak maximum principle, $\begin{cases} v(x,t) \leq 0 \text{ and } -v(x,t) \leq 0 \\ \Rightarrow v(x,t) = 0, \ 0 \leq x \leq 1, \ 0 \leq t \leq T \end{cases}$

As T is arbitrary, $v(x,t) = 0$, $u(x,t) = 2$, $0 \leq x \leq 1$, $t \geq 0$

Problem 4:(25pts)
1.)(13pts) We consider initial and boundary value the problem

$$u_{tt} + u_t - u_{xx} = 0, \quad 0 < x < L, \ t > 0 \tag{15}$$
$$u(0,t) = f(t), \ u(L,t) = g(t), \quad t \geq 0 \tag{16}$$
$$u(x,0) = \psi(x), \ u_t(x,0) = \varphi(x), \quad 0 \leq x \leq L, \tag{17}$$

where f, g, ψ, φ are continuous functions.
Show that the problem has a unique solution.
Hint: You may estimate the quantity $\frac{d}{dt}\int_0^L (w_t^2 + w_x^2)dx$, where w is the difference of two solutions of the problem.
2.)(12pts) Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the initial and boundary value problem

$$u_t - \Delta u + u^5 = 0, \qquad x \in \Omega, \ t > 0, \tag{18}$$
$$u(x,t) = 0, \qquad x \in \partial\Omega, \ t \geq 0, \tag{19}$$
$$u(x,0) = u_0(x), \qquad x \in \Omega. \tag{20}$$

Show that, if $\int_\Omega (u_0(x))^2 dx \leq 1$, then

$$\int_\Omega (u(x,t))^2 dx \leq 1, \quad \forall t \geq 0. \tag{21}$$

Solution:

1) Let $w = u_1 - u_2$, where u_1 and u_2 are two solutions of (15)-(17)

We have $w_{tt} + w_t - w_{xx} = 0$ (7), $w(0,t) = w(L,t) = w(x,0) = w_t(x,0) = 0$

We multiply (7) by w_t, and integrate on x from 0 to L, to find

$$\frac{1}{2}\frac{d}{dt}\int_0^L w_t^2 dx + \underbrace{\int_0^L w_t^2 dx}_{\geq 0} + \frac{1}{2}\int_0^L w_x^2 dx + \underbrace{\left[w_x w_t\right]_0^L}_{=0} = 0$$

$$\Rightarrow \frac{d}{dt}\int_0^L (w_t^2 + w_x^2)\,dx \leq 0 \Rightarrow \int_0^L \left[w_t^2(x,t) + w_x^2(x,t)\right]dx \leq \underbrace{\int_0^L \left[w_t^2(x,0) + w_x^2(x,0)\right]dx}_{=0}$$

$$\Rightarrow w_t(x,t) = 0, \ w_x(x,t) = 0 \Rightarrow w(x,t) = 0, \ \forall x \in [0,L], \ t \geq 0.$$

2) We multiply (18) by u, integrate over Ω, and we find

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \underbrace{\int_\Omega (\nabla u)^2 dx}_{\geq 0} + \underbrace{\int_{\partial\Omega} \frac{\partial u}{\partial n} u \, dx}_{=0} + \underbrace{\int_\Omega u^6 dx}_{\geq 0} = 0$$

(here we used Green identity)

$$\Rightarrow \frac{d}{dt}\int_\Omega u^2 dx \leq 0 \Rightarrow \int_\Omega u^2(x,t)\,dx \leq \int_\Omega u^2(x,0)\,dx, \ \forall t \geq 0$$

If $\int_\Omega u^2(x,0)dx \leq 1 \Rightarrow \int_\Omega u^2(x,t)\,dx \leq 1, \ \forall t \geq 0$

Problem 5:

1.)(5pts) Let $u \in \mathcal{C}^2(\mathbb{R})$ be a solution of the equation

$$u_{xx} + 4u + 4u_x + u_{yy} = 0. \tag{22}$$

Find a twice differentiable function f on $\mathbb{R}$ such that $f(0) = 1$, $f'(0) = 2$ and such that $v(x,y) = f(x)u(x,y)$ is solution of the Laplace equation

$$v_{xx} + v_{yy} = 0. \tag{23}$$

2.)(20pts) Solve the Laplace problem

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < 1,\ 0 < y < 1, \tag{24}$$

$$u_x(0,y) = u_x(1,y) = 0, \quad 0 \le y \le 1, \tag{25}$$

$$u(x,0) = 0, \quad u(x,1) = g(x), \quad 0 \le x \le 1. \tag{26}$$

Solution:

1.) $V_x = f'u + fu_x$; $V_{xx} = f''u + 2f'u_x + fu_{xx}$; $V_{yy} = fu_{yy}$

$V_{xx} + V_{yy} = 0 \Rightarrow f''u + 2f'u_x + fu_{xx} + fu_{yy} = 0$

We can set $f'' = 4f$, $f' = 2f$, $f(0) = 1$, $f'(0) = 2 \Rightarrow \underline{f(x) = e^{2x}}$

2.) Let $u = XY$. Thus, $X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$

$u_x(0,y) = 0 \Rightarrow X'(0) = 0$

$u_x(1,y) = 0 \Rightarrow X'(1) = 0$

$u(x,0) = 0 \Rightarrow Y(0) = 0$

- If $\lambda = 0$, then $X'' = 0$, $X = c_1 x + c_2$, $x' = c_1$; $\left\{ X'(0) = X'(1) = 0 \Rightarrow c_1 = 0 \right.$

  $Y'' = 0$, $Y = c_3 y + c_4$. $Y(0) = 0 \Rightarrow c_4 = 0$

  $\Rightarrow u = cy$

- If $\lambda = \omega^2$, then $X'' - \omega^2 X = 0 \Rightarrow X = c_1 e^{\omega x} + c_2 e^{-\omega x}$, $X' = c_1 \omega e^{\omega x} - c_2 \omega e^{-\omega x}$ $\left\{ X'(0) = X'(1) = 0 \right\} \Rightarrow c_1 = c_2 = 0$

- if $\lambda = -\omega^2$, then $X'' + \omega^2 X = 0$, $X = c_1 \cos \omega x + c_2 \sin \omega x$

  $X' = -\omega c_1 \sin \omega x + c_2 \omega \cos \omega x$

  $X'(0) = X'(1) = 0 \Rightarrow c_2 = 0$, $\omega = n\pi$

  $Y'' - \omega^2 Y = 0$, $Y = d_1 \cosh \omega y + d_2 \sinh \omega y$

  $Y(0) = 0 \Rightarrow d_1 = 0$

Thus $u(x,y) = \sum_{n=1}^{\infty} A_n \cos(n\pi x) \sinh(n\pi y) + A_0 y$

$\Rightarrow g(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \sinh n\pi \Rightarrow \left\{ \begin{array}{l} A_0 = \int_0^1 g(x)\,dx \\ A_n = \frac{2}{\sinh n\pi} \int_0^1 g(x) \cos(n\pi x)\,dx \end{array} \right.$