King Fahd University of Petroleum and Minerals
Department of Mathematics
PDE Comprehensive Exam
The First Semester of 2022-2023 (221)
Time Allowed: 120mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		25
2		25
3		25
4		25
5		25
Total		25 $\times$4 =100

Solve 4 problems of your choice.

Remark: **Only 4 problems will be taking into account, even if you solve the five problems.**

Problem 1:(25pts)

1.)(13pts) Use the method of characteristics to solve the Cauchy problem:

$$u_x + e^{-y}u_y = e^y, \quad u(0,y) = e^{2y}, \quad x > 0, \ y \in \mathbb{R}.$$

2.)(12pts) Find the general solution $w(\xi,\eta)$ of the hyperbolic PDE

$$w_{\xi\eta} + w_\xi = 2\xi.$$

Solution

1) $\frac{dx}{dt} = 1 \Rightarrow \int dx = \int dt \Rightarrow x = t + C_1$

$\frac{dy}{dt} = e^{-y} \Rightarrow \int e^{y} dy = \int dt \Rightarrow e^{y} = t + C_2$

$\frac{du}{dt} = e^{y} \Rightarrow \int du = \int (t + C_2)\, dt \Rightarrow u = \frac{t^2}{2} + C_2 t + C_3$

At $t=0$, if $y=s$, then $x=0$ and $u = e^{2s}$

Thus, $C_1 = 0$, $C_2 = e^{s}$ and $C_3 = e^{2s}$.

This implies $x = t$, $e^{y} = t + e^{s}$ and $u = \frac{t^2}{2} + te^{s} + e^{2s}$

$\Rightarrow \boxed{u = \frac{x^2}{2} + x(e^{y} - x) + (e^{y} - x)^2}$ or $\boxed{u = \frac{x^2}{2} - xe^{y} + e^{2y}}$

2) $w_{\xi\eta} + w_\xi = 2\xi$ (1)

Multiply (1) by e^{η} $\Rightarrow \frac{\partial}{\partial \eta}(e^{\eta} w_\xi) = 2\xi e^{\eta}$ (2)

We integrate (2) with respect to η to find

$e^{\eta} w_\xi = 2\xi e^{\eta} + f(\xi)$

$\Rightarrow w_\xi = 2\xi + e^{-\eta} f(\xi)$ (3)

We integrate (3) with respect to ξ to find

$\boxed{w(\xi,\eta) = \xi^2 + e^{-\eta} G(\xi) + K(\eta)}$,

where G and K are C^2 and C^1 functions, respectively.

Problem 2:(25pts)

1.)(**10**pts) Find the solution of the 1D Cauchy problem

$$u_{tt} - 9u_{xx} = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty), \tag{1}$$
$$u(x,0) = x^2, \; u_t(x,0) = \cos^2 x, \; x \in \mathbb{R}. \tag{2}$$

2.)(**15**pts) Solve the 2D Cauchy problem

$$u_{tt} = u_{xx} + u_{yy}, \quad (x,y) \in \mathbb{R}^2, \; t > 0, \tag{3}$$
$$u(x,y,0) = 0, \quad (x,y) \in \mathbb{R}^2, \tag{4}$$
$$u_t(x,y,0) = y, \quad (x,y) \in \mathbb{R}^2. \tag{5}$$

Solution

1.) D'Alembert formula gives

$$u(x,t) = \frac{(x+3t)^2 + (x-3t)^2}{2} + \frac{1}{9}\int_{x-3t}^{x+3t} \cos^2 s \, ds \; ; \quad \cos^2 s = \frac{1+\cos 2s}{2}$$

$$= x^2 + 9t^2 + \frac{t}{3} + \frac{1}{36}\left[\sin 2(x+3t) - \sin 2(x-3t)\right]$$

2.) Kirchhoff's formula gives

$$u(x,y,t) = \frac{1}{2\pi}\iint_{D_t} \frac{\eta}{\sqrt{t^2-(\xi-x)^2-(\eta-y)^2}}\, d\xi\, d\eta,$$

where $D_t = \{(\xi,\eta) \in \mathbb{R}^2, \; (\xi-x)^2+(\eta-y)^2 \le t^2\}$

Consider the polar coordinates

$$\begin{cases} \xi = x + r\cos\theta \\ \eta = y + r\sin\theta \end{cases} \Rightarrow d\xi\, d\eta = r\, dr\, d\theta, \quad 0 \le r \le t, \; 0 \le \theta \le 2\pi$$

$$\Rightarrow u(x,y,t) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^t \frac{r(y + r\sin\theta)}{\sqrt{t^2-r^2}}\, dr\, d\theta$$

$$= \frac{y\, 2\pi}{2\pi} \underbrace{\int_0^t \frac{r}{\sqrt{t^2-r^2}}\, dr}_{\left[-(t^2-r^2)^{\frac{1}{2}}\right]_0^t} + \frac{1}{2\pi}\underbrace{\left(\int_0^{2\pi} \sin\theta\, d\theta\right)}_{=0}\left(\int_0^t \frac{r^2}{\sqrt{t^2-r^2}}\, dr\right)$$

$$\boxed{u(x,y,t) = yt}$$

Problem 3:(25pts)

1.)(15 pts) Solve the initial and boundary value problem

$$u_t = 4u_{xx}, \quad x \geq 0,\ t \geq 0, \tag{6}$$

$$u(x,0) = \begin{cases} 1, & \text{if } \ x \in [0,\pi], \\ 0, & \text{elsewhere,} \end{cases} \tag{7}$$

$$u(0,t) = 0, \quad \forall t \geq 0. \tag{8}$$

2.(10 pts) Assume $u(x,t) = \sum_{n=1}^{\infty} 2A_n \sin(n\pi x) e^{-\pi^2 n^2 t}$ is the solution the IBVP

$$u_t = u_{xx}, \quad 0 < x < 1,\ t > 0, \tag{9}$$

$$u(x,0) = 1, \quad 0 \leq x \leq 1. \tag{10}$$

Compute explicitly A_n.

<u>Solution</u>

1.) $u = TX \Rightarrow T'X = 4TX'' \Rightarrow \frac{T'}{4T} = \frac{X''}{X} = \lambda$

- If $\lambda = 0$, $X'' = 0$, $X = c_1 x + c_2$. X bounded $\Rightarrow c_1 = 0$; $X(0) = 0 \Rightarrow c_2 = 0$ $\Rightarrow u = 0$
- If $\lambda = w^2$, $X'' - w^2 X = 0$, $X = c_1 e^{wx} + c_2 e^{-wx}$. X bounded $\Rightarrow c_1 = c_2 = 0$. $X(0) = 0$, $w > 0$.
- If $\lambda = -w^2$, $X'' + w^2 X = 0$, $X = c_1 \cos wx + c_2 \sin wx$. $X(0) = 0 \Rightarrow c_1 = 0$

$T' + 4w^2 T = 0$, $T = c e^{-4w^2 t}$ $\Rightarrow u(x,t) = B_w \int_0^{\infty} \sin wx \, e^{-4w^2 t} dw$

$u(x,0) = B_w \int_0^{\infty} \sin wx \, dw \Rightarrow B_w = \frac{2}{\pi} \int_0^{\infty} u(\xi,0) \sin w\xi \, d\xi = \frac{2(1-\cos \pi w)}{\pi w}$

2.) At $t = 0$, we have $\sum_{n=1}^{\infty} 2A_n \sin n\pi x = 1$ (1)

We multiply (1) by $\sin m\pi x \Rightarrow \sum_{n=1}^{\infty} 2A_n \underbrace{\int_0^1 \sin n\pi x \sin m\pi x}_{I} = \underbrace{\int_0^1 \sin m\pi x}_{J}$

$I = \frac{1}{2} \int_0^1 [\cos(n-m)\pi x - \cos(n+m)\pi x] \, dx = \begin{cases} 0, & n \neq m \\ \frac{1}{2}, & n = m \end{cases}$

$J = \frac{1}{m\pi}(1 - \cos m\pi) = \frac{1-(-1)^m}{m\pi}$

$\Rightarrow \boxed{A_m = \frac{1-(-1)^m}{m\pi}}$

Problem 4:(25pts)

1.(10 pts) Let Ω be a bounded domain of $\mathbb{R}^3$, with smooth boundary $\partial\Omega$. Consider the boundary value problem

$$-\Delta u + u = \lambda u, \quad \text{on } \in \Omega, \tag{11}$$
$$u = 0, \quad \text{on } \in \partial\Omega, \tag{12}$$

where $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Prove that if this problem admits a non trivial solution u (that is, $u \neq 0$), then $\lambda \geq 1$.

2.(15 pts) Use the weak maximum principle to show that any continuous solution u of the initial and boundary value problem

$$u_t = u_{xx} + 2, \quad 0 < x < 1, \ t > 0 \tag{13}$$
$$u(0,t) = u(1,t) = 2t \quad t \geq 0, \tag{14}$$
$$u(x,0) = 0, \quad 0 \leq x \leq 1 \tag{15}$$

satisfies the estimate $u(x,t) \leq 2t$ for all $0 \leq x \leq 1$ and $t \geq 0$.

Note: Use the weak maximum principle **only** to answer the question. Any other different method to answer the question will be given a grade of zero.

Solution

1) $-\Delta u + u = \lambda u \qquad (1)$

We multiply (1) by u and integrate over Ω, to find

$$\underbrace{-\int_\Omega \Delta u\, u\, dx}_{} + \int_\Omega u^2 dx = \lambda \int_\Omega u^2 dx$$

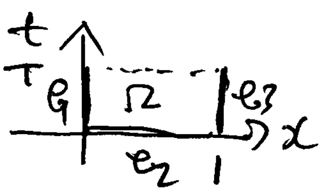
Green identity : $\int_\Omega |\nabla u|^2 dx + \underbrace{\int_{\partial\Omega} \frac{\partial u}{\partial n} u\, dx}_{=0}$ $\Rightarrow \lambda = \dfrac{\int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^2 dx}{\int_\Omega |u|^2 dx} \geq 1$

2) Let $V = u - 2t$

The function v satisfies the problem

$$\begin{cases} V_t = V_{xx}, & 0 \leq x < 1, \ t > 0 \\ V(0,t) = V(1,t) = 0, & t \geq 0 \\ V(x,0) = 0 & 0 \leq x \leq 1 \end{cases}$$

Let $T > 0$ arbitrary and $\Omega_T = [0,1] \times [0,T]$



- $V(x,t) = 0$ on $e_1 \cup e_2 \cup e_3$
- V is continuous on Ω

By the weak maximum principle, $V(x,t) \leq 0$, for $0 \leq x \leq 1$, $0 \leq t \leq T$

As T is arbitrary $\Rightarrow$ $V(x,t) \leq 0$, $0 \leq x \leq 1$, $t \geq 0$

$\Rightarrow$ $u(x,t) \leq 2t$, $0 \leq x \leq 1$, $t \geq 0$

Problem 5:(25pts)

1.)(5pts) Consider the problem

$$u_t + \mu(t)u = \Delta u, \quad x \in \mathbb{R}^3,\ t > 0. \tag{16}$$

Write the differential equation satisfied by the function v, where $v(x,t) = u(x,t)e^{\int \mu(t)dt}$.

2.)(20pts) Solve the 2D Neumann problem

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < 1,\ 0 < y < 2, \tag{17}$$

$$u_x(0,y) = u_x(1,y) = 0, \quad 0 \le y \le 2, \tag{18}$$

$$u_y(x,0) = 0, \quad u_y(x,2) = g(x), \quad 0 \le x \le 1. \tag{19}$$

Solution

1.) $V = ue^{\int \mu(t)dt}$, $V_t = U_t e^{\int \mu(t)dt} + \mu(t) u e^{\int \mu(t) dt} \Rightarrow U_t = V_t e^{-\int \mu} - \mu(t)u$

$\Delta V = e$ $\quad \Delta V = e^{\int \mu(t)dt} \Delta u \Rightarrow \Delta u = e^{-\int \mu(t)dt} \Delta V$

$\Rightarrow V_t e^{-\int \mu} - \mu(t)u + \mu(t)u = e^{-\int \mu} \Delta V \Rightarrow \boxed{V_t = \Delta V}$

2.) $u = XY$, $X''Y + XY'' = 0$, $\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$, $X'(0) = X'(1) = 0$

$Y'(0) = 0$

- If $\lambda = 0$, $X'' = 0$, $X = c_1 x + c_2$, $X' = c_1 \Rightarrow c_1 = 0$
  $Y'' = 0$, $Y = c_3 y + c_4$, $Y' = c_3 \Rightarrow c_3 = 0$ $\Big\}\ U = c$
- If $\lambda = \omega^2$, $X'' - \omega^2 X = 0$, $X = c_1 e^{\omega x} + c_2 e^{-\omega x}$, $X' = c_1 \omega e^{\omega x} - c_2 \omega e^{-\omega x}$
  $X'(0) = 0 \Rightarrow c_1 \omega - c_2 \omega = 0$
  $X'(1) = 0 \Rightarrow c_1 \omega e^{\omega} - c_2 \omega e^{-\omega} = 0$ $\Big\} \Rightarrow c_1 = c_2 = 0$
- If $\lambda = -\omega^2$, $X'' + \omega^2 X = 0$, $X = c_1 \cos \omega x + c_2 \sin \omega x$, $X' = -c_1 \omega \sin \omega x + c_2 \omega \cos \omega x$
  $X'(0) = 0 \Rightarrow c_2 \omega = 0 \Rightarrow c_2 = 0$
  $X'(1) = 0 \Rightarrow \sin \omega = 0$, $\omega = n\pi$
  $Y'' - n^2\pi^2 Y = 0 \Rightarrow Y = c_1 e^{n\pi y} + c_2 e^{-n\pi y}$
  $Y' = c_1 n\pi e^{n\pi y} - c_2 n\pi e^{-n\pi y}$;
  $Y'(0) = 0 \Rightarrow c_1 = c_2$

$\Rightarrow \quad u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \cosh(n\pi y)$

$\sum_{n=1}^{\infty} n\pi A_n \cos(n\pi x) \sinh(n\pi 2) = g(x) \Rightarrow A_n = \frac{2}{n\pi \sinh(2\pi n)} \int_0^1 g(\xi) \cos(n\pi \xi)\, d\xi$, $n \neq 0$