King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Numerical Analysis of Ordinary Differential Equations
Comprehensive Exam

Name:	**Key Solution**
ID :	

Q		Points
1		**20**
2		**20**
3		**20**
4		**10**
5		**20**
6		**20**
Total		**100**

Q1	Consider the Runge-Kutta method with tableau given below

$$\begin{array}{c|cc} 1/4 & 7/24 & -1/24 \\ 3/4 & 13/24 & 5/24 \\ \hline & 1/2 & 1/2 \end{array}$$

a) Find the stability function of the method.

b) Is the method A-stable?

In MATH571 textbook page 231[Butcher, John Charles. *Numerical methods for ordinary differential equations*. John Wiley & Sons, 2016.],

Theorem 351B *A Runge–Kutta method with stability function $R(z) = N(z)/D(z)$ is A-stable if and only if (a) all poles of R (that is, all zeros of D) are in the right half-plane and (b) $E(y) \geq 0$, for all real y.*

1(a) $\dfrac{c \mid A}{\mid b^T}$ Stability function is

$$R(z) = \frac{\det(I + z(1b^T - A))}{\det(I - zA)}$$

$$N(z) = \left| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + z\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} - \begin{bmatrix} 7/24 & -1/24 \\ 13/24 & 5/24 \end{bmatrix} \right) \right|$$

$$= \begin{vmatrix} 1 + \frac{5}{24}z & \frac{13}{24}z \\ -\frac{1}{24}z & 1 + \frac{7}{24}z \end{vmatrix} = 1 + \frac{1}{2}z + \frac{1}{12}z^2$$

$$D(z) = |I - zA| = \begin{vmatrix} 1 - \frac{7}{24}z & \frac{1}{24}z \\ -\frac{13}{24}z & 1 - \frac{5}{24}z \end{vmatrix}$$

$$= 1 - \frac{1}{2}z + \frac{1}{12}z^2$$

$$R(z) = \left(1 + \frac{1}{2}z + \frac{1}{12}z^2\right) \Big/ \left(1 - \frac{1}{2}z + \frac{1}{12}z^2\right)$$

(b) $D(z) = 0 \Rightarrow z = 3 \pm i\sqrt{3}$

real part of zeros lies in the right half-plane

$$E(y) = D(iy)D(-iy) - N(iy)N(-iy)$$

$$= \left(1 - \frac{iy}{2} - \frac{y^2}{12}\right)\left(1 + \frac{iy}{2} - \frac{y^2}{12}\right) - \left(1 + \frac{iy}{2} - \frac{y^2}{12}\right)\left(1 - \frac{iy}{2} - \frac{y^2}{12}\right)$$

$$= 0$$

$\Rightarrow$ the method is A-stable

Q2	Consider the linear multistep method $y_n = y_{n-1} + 2hf(x_{n-1}, y_{n-1}) - hf(x_{n-2}, y_{n-2})$ Determine if the method is consistent and stable. If it is consistent, then find the order.

(1) Consistent Method:

First let us start by finding the truncation error:

$$\begin{aligned}\boldsymbol{\tau(h)} &= y(x_n) - y(x_n - h) - 2hf\big(x_n - h, y(x_n - h)\big) + hf\big(x_n - 2h\,, y(x_n - 2h)\big)\\ &= y(x_n) - y(x_n - h) - 2h\,y'(x_n - h) + h\,y'(x_n - 2h)\\ &= y(x_n) - \left[y(x_n) - h\,y'(x_n) + \frac{h^2}{2}y''(x_n) + \cdots\right] - 2h\,[y'(x_n) - hy''(x_n) + \cdots] + h\,[y'(x_n) - 2hy''(x_n) + \cdots]\\ &= -\frac{h^2}{2}\,y''(x_n) + \cdots\end{aligned}$$

So $\boldsymbol{\tau(h) \to 0}$ $\boldsymbol{as\ h \to 0}$. Hence, the method is consistent and the order of the method is 1.

OR

In MATH571 textbook page 107 [Butcher, John Charles. *Numerical methods for ordinary differential equations*. John Wiley & Sons, 2016.], General form of linear multistep methods

$$\begin{aligned}y_n = \alpha_1 y_{n-1} + \alpha_2 y_{n-2} + \cdots + \alpha_k y_{n-k}\\ + h\big(\beta_0 f(x_n, y_n) + \beta_1 f(x_{n-1}, y_{n-1}) + \beta_2 f(x_{n-2}, y_{n-2})\\ + \cdots + \beta_k f(x_{n-k}, y_{n-k})\big).\end{aligned}$$

$\boldsymbol{k = 2, \alpha_1 = 1, \alpha_2 = 0\,, \beta_0 = 0, \beta_1 = 2\,, \beta_2 = -1}$

Associate pair of polynomials

$\boldsymbol{\alpha(z) = 1 - z,} \qquad \boldsymbol{\beta(z) = 2z - z^2}$

A **'consistent method'** is a method that satisfies the condition that $\beta(1) + \alpha'(1) = 0$, in addition to satisfying the **preconsistency** condition $\alpha(1) = 0$.

$\alpha(1) = 0$ and $\alpha'(z) = -1$. Moreover, we have $\beta(1) + \alpha'(1) = (2 - 1) + (-1) = 0$. Hence the method is consistent.

> **Theorem 410B** *A linear multistep method $[\alpha, \beta]$ has order p (or higher) if and only if*
>
> $$\alpha(\exp(z)) + z\beta(\exp(z)) = O(z^{p+1}).$$

$$\boldsymbol{\alpha(e^z) + z\beta(e^z) = 1 - e^z + z\,(\,2e^z - e^{2z}\,) = 1 - e^z + 2\,z\,e^z - z\,e^{2z}}$$

$$\boldsymbol{= 1 - \left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots\right) + 2z\left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots\right) - z\left(1 + 2z + 2z^2 + \frac{4}{3}z^3 + \cdots\right)}$$

$\boldsymbol{= -\frac{1}{2}z^2 + \cdots = O(z^2)}$. **Hence the order of the method is 1.**

(1) Stable Method:

From [Burden RL, Faires JD, Burden AM. Numerical analysis. Cengage learning; 2015.] page 346

Definition 5.22 Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ denote the (not necessarily distinct) roots of the characteristic equation

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - \cdots - a_1\lambda - a_0 = 0$$

associated with the multistep difference method

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad \ldots, \quad w_{m-1} = \alpha_{m-1}$$

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} + hF(t_i, h, w_{i+1}, w_i, \ldots, w_{i+1-m}).$$

If $|\lambda_i| \leq 1$, for each $i = 1, 2, \ldots, m$, and all roots with absolute value 1 are simple roots, then the difference method is said to satisfy the **root condition**. ■

Definition 5.23

(i) Methods that satisfy the root condition and have $\lambda = 1$ as the only root of the characteristic equation with magnitude one are called **strongly stable**.

(ii) Methods that satisfy the root condition and have more than one distinct root with magnitude one are called **weakly stable**.

(iii) Methods that do not satisfy the root condition are called **unstable**. ■

The method can be rewritten as : $y_{n+1} = y_n + 2hf(x_n\,, y_n) - h\,f(x_{n-1\,,}\, y_{n-1})$
In this case, $m = 2, a_1 = 1, a_2 = 0$. So the characteristic equation is $P(\lambda) = \lambda^2 - \lambda = \lambda\,(\lambda - 1)$.
This polynomial has roots $\lambda_1 = 0, \lambda_2 = 1$. Hence, it satisfies the root condition and is strongly stable.

Q3	Prove the following Theorem: Theorem: Let f be a continuous function and satisfies a Lipschitz condition for $[a,b]\times R^1$. Let $y\in C^2[a,b]$ be the solution of $\begin{cases} y' = f(x,y), & x\in[a,b], \; y\in R^1 \\ y(a) = y_0 \end{cases}$ Then, $\exists$ a constant $K>0$ such that e_k, the global error in Euler's method, satisfies $\lvert e_k\rvert \le Kh, \quad k = 0,1,2,\ldots,N$

PROOF:

Taylor series gives:

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{1}{2}h^2\, y''(\xi_i)$$

Using the notation $y(t_i) = y_i$:

$$y_{i+1} = y_i + h\, f(t_i,\, y_i) + \frac{1}{2}h^2\, y''(\xi_i) \qquad \textbf{(3)}$$

Euler's formula :

$$w_{i+1} = w_i + hf(t_i, w_i) \qquad \textbf{(4)}$$

Subtract (4) from (3):

$$y_{i+1} - w_{i+1} = y_i - w_i + h[f(t_i, y_i) - f(t_i, w_i)] + \frac{1}{2}h^2\, y''(\xi_i)$$

Take absolute value and use triangle inequality

$$|y_{i+1} - w_{i+1}| \le |y_i - w_i| + h|f(t_i, y_i) - f(t_i, w_i)| + \frac{1}{2}h^2\, |y''(\xi_i)|$$

Using Lipschitz condition and the bound $|\, y''(t)\, | \le M$ give:

$$|y_{i+1} - w_{i+1}| \le |y_i - w_i| + h\, L\, |y_i - w_i| + \frac{h^2 M}{2}$$

Using the notation $E_i = |y_i - w_i| = |\, e_i\, |$:

$$E_{i+1} \le E_i + h\, L\, E_i + \frac{h^2 M}{2}$$

$$E_{i+1} \le (1 + hL)E_i + \frac{h^2 M}{2} \qquad \textbf{(5)}$$

Let $i = 0,\ 1,\ 2,\ \cdots$ in equation (5)

$$i = 0 \rightarrow \quad E_1 \le (1 + hL)E_0 + \frac{h^2 M}{2} \quad \le \frac{h^2 M}{2} \qquad e_0 = |y_0 - w_0| = 0$$

$$i = 1 \rightarrow \quad E_2 \le (1 + hL)E_1 + \frac{h^2 M}{2} \quad \le (1 + hL)\frac{h^2 M}{2} + \frac{h^2 M}{2}$$

$$i = 2 \rightarrow \quad E_3 \le (1 + hL)E_2 + \frac{h^2 M}{2} \quad \le (1 + hL)^2\frac{h^2 M}{2} + (1 + hL)\frac{h^2 M}{2} + \frac{h^2 M}{2}$$

Clearly by induction for the case $i = k$ we have :

$$E_k \le (1 + hL)E_{k-1} + \frac{h^2 M}{2}$$

$$\le (1 + hL)^{k-1}\frac{h^2 M}{2} + (1 + hL)^{k-2}\frac{h^2 M}{2} + \cdots + \frac{h^2 M}{2}$$

$$\le [(1 + hL)^{k-1} + (1 + hL)^{k-2} + \cdots + 1]\frac{h^2 M}{2}$$

(geometric series with ratio r=(1+hL))

$$\le \left[\frac{1 - (1 + hL)^k}{1 - (1 + hL)}\right]\frac{h^2 M}{2} \qquad 1 + r + r^2 + \cdots + r^i = \frac{1 - r^{i+1}}{1 - r}$$

$$\le [(1 + hL)^k - 1]\frac{h\,M}{2\,L}$$

$$\le \left[\left(e^{h\,L}\right)^k - 1\right]\frac{h\,M}{2\,L}$$

see remark: $1 + h\,L \le e^{h\,L}$

$$\le [e^{L\,k\,h} - 1]\frac{h\,M}{2\,L}$$

$$\le [e^{\,L\,(t_k - a)} - 1]\frac{h\,M}{2\,L} \qquad t_k = a + kh$$

$$\le [e^{\,L\,(b - a)} - 1]\frac{h\,M}{2\,L} \qquad t_k \le b\,, for\ all\ k$$

Hence, we have the error bound:

$$E_k \le [e^{\,L\,(b - a)} - 1]\frac{h\,M}{2\,L}$$

$$|e_k| \le K\,h \qquad \text{where} \qquad K = \left(\frac{[e^{\,L\,(b - a)} - 1]\,M}{2\,L}\right)$$

$\blacksquare$

Remark

a Taylor series gives:

Let $x = h\,L$: $e^x = 1 + x + \underbrace{\frac{1}{2}x^2\,e^{\xi}}_{\text{positive}}$

$$e^{h\,L} = 1 + h\,L + \frac{1}{2}(h\,L)^2\,e^{\xi} \ \ge 1 + h\,L$$

$$e^{h\,L} \ge 1 + h\,L$$

Q4	Prove that: Explicit Runge–Kutta methods can never be A-stable

For any Explicit Runge–Kutta method, defined by the tableau

$$\begin{array}{c|c} c & A \\ \hline & b^{\top} \end{array}$$

The matrix A is lower triangular (all diagonal entries and above the diagonal are zeros). Now the stability function is given by

$$R(z) = \frac{\det\left(I + z(\mathbf{1}b^{\top} - A)\right)}{\det(I - zA)}.$$

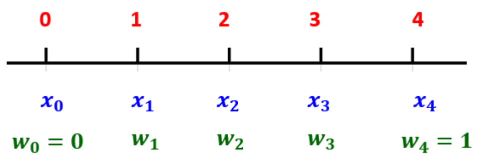
Note that $det(I - zA) = 1$. (i.e) $R(z) = det(I + z(1\, b\text{^}T - A))$ (is a polynomial)

R(z) is polynomial. R(x) is a polynomial where x is real number. $\lim_{x \to \pm\infty} |R(x)| = +\infty$. which implies that the method is not A-stable

Q5 Solve the given boundary value problem using the finite difference method

$$u'' + 6u = x \quad \text{with boundary conditions} \quad u(0) = 0, u(4) = 1.$$

Suppose the finite difference approximation for the second-order derivative $u'' \approx (u_{i+1} - 2u_i + u_{i-1})/h^2$ and the given interval is divided into four equal subintervals. (Just write the linear system. Don't solve the system)



The mesh size is $h = \frac{b-a}{4} = \frac{4-0}{4} = 1$

$$u''(x) + 6\,u(x) = x$$

We replace the second derivative by central difference

$$\frac{w_{i+1} - 2\,w_i + w_{i-1}}{h^2} + 6\,w_i = x_i \quad \text{for } i = 1, 2, 3$$

$$(w_{i+1} - 2\,w_i + w_{i-1}) + 6\,w_i = x_i \quad \text{for } i = 1, 2, 3$$

$$w_{i-1} + 4\,w_i + w_{i+1} = x_i \quad \text{for } i = 1, 2, 3$$

for $i = 1$ $\quad w_0 + 4\,w_1 + w_2 = x_1$

for $i = 2$ $\quad w_1 + 4\,w_2 + w_3 = x_2$

for $i = 3$ $\quad w_2 + 4\,w_3 + w_4 = x_3$

Use the boundary conditions

$$0 + 4\,w_1 + w_2 = 1$$

$$w_1 + 4\,w_2 + w_3 = 2$$

$$w_2 + 4\,w_3 + 1 = 3$$

It is a linear system of three equations in three unknowns w_1, w_2 *and* w_3

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Q6	Consider the following three problems

(BVP)

$$u''-3x^2u'+3xu=5$$
$$u(0)=1, \qquad u(1)=2$$

(IVP-1)

$$u_1''-3x^2u_1'+3xu_1=5$$
$$u_1(0)=1, \qquad u_1'(0)=0$$

(IVP-2)

$$u_2''-3x^2u_2'+3xu_2=5$$
$$u_2(0)=1, \qquad u_2'(0)=1$$

The numerical solutions of the (IVP-1) and (IVP-2) are given in column(1) and column(2), respectively. Use shooting method to find the numerical solution of the (BVP) then fill column(3).

x	$u_1(x)$	$u_2(x)$	$u(x)$
0	1	1	
0.2	1.095654	1.294856	
0.4	1.357517	1.745005	
0.6	1.719056	2.25898	
0.8	2.074545	2.702588	
1	2.315669	2.955164	

The shooting method for linear equations is based on the replacement of the linear boundary- value problem by two initial-value problems.
Since the differential equation in (BVP) is a linear ode.
Now, we can write the solution of (BVP) as a linear combination of the solutions of (IVP1) and (IVP2).

$$w(x) = \lambda\, u_1(x) + (1-\lambda)u_2(x)$$

$$w(x) = \lambda u_1(x) + (1-\lambda)u_2(x)$$
$$w(0) = \alpha$$

We want

$$w(1) = \lambda u_1(1) + (1-\lambda)u_2(1) = \beta$$

Solve: $\lambda = \dfrac{\beta - u_1(1)}{u_1(1) - u_2(1)}$ $\qquad \lambda = 1.49362242$

x	$u_1(x)$	$u_2(x)$	$u(x)$
0	1	1	**1**
0.2	1.095654	1.294856	**0.997323426691160**
0.4	1.357517	1.745005	**1.166244235719040**
0.6	1.719056	2.25898	**1.452537408503920**
0.8	2.074545	2.702588	**1.764528894475940**
1	2.315669	2.955164	**2.0**