Department of Mathematics and Statistics, KFUPM
Comprehensive Exam, Math 571, 20 Jan, 2021, Duration: 150 mins

Solution

Problem 1(20 points) : Consider the initial value problem

$$y' = x\sin(y) \quad \text{for } x \in [0, 2], \quad y(0) = \pi/2 \tag{1}$$

a) Show that (1) has a unique solution $y \in C^1[0, b]$ for some $b > 0$.
b) Define the one-step Explicit Euler scheme and show that the global error is $O(h)$.
c) Define the one-step implicit Euler scheme and show that the truncation error T_n is $O(h)$.

Solution:
a) Using Picar's Theorem the problem (1) has a unique solution if $f(x, y) = x\sin(y)$ continuous in its first variable and satisfies a Lipschitz condition in its second variable. Clearly f is a continuous function with respect to $x \in [0, 2]$. Since $\frac{\partial f}{\partial y} = x\cos(y)$, then

$$|f(x, y_1) - f(x, y_2)| \leq |x\cos y||y_1 - y_2| \leq 2|y_1 - y_2|$$

That is f is Lipschitz continuous with Lipschitz constant $L = 2$. Thus, (1) has a unique solution.
for part (a) 3 points
b) The Explicit Euler method is given by dividing the interval $[0, 2]$ into N sub- intervals using mesh points $x_n = nh$ for $0 \leq n \leq N$ with $h = 2/N$. If we denote the by y_n the approximate solution of $y(x)$ at x_n, then the scheme is given by

$$y_0 = y(0) \tag{2}$$

$$y_{n+1} = y_n + hf(x_n, y_n), \quad 0 \leq n \leq N - 1 \quad \text{2 points} \tag{3}$$

To estimate the global error, we need bound for the truncation error which is given by the following formula

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - f(x_n, y(x_n)) \tag{4}$$

By noting that $f(x_n, y(x_n)) = y'(x_n)$ and applying Taylor's Theorem, it follows from (4) that there exists $\xi_n \in (x_n, x_{n+1})$ such that

$$|T_n| = \frac{1}{2}h|y''(\xi_n)| \tag{5}$$

Note that

$$y''(x) = \sin(y) + x\cos(y)y'(x) = \sin(y) + x^2\cos(y)\sin(y)$$

So that $|y''(x)| \leq 1 + x^2 \leq 1 + 4 = 5$ (1 point) and then

$$|T_n| \leq \frac{5h}{2} \qquad \text{(2 points)}$$

Since from the definition of Euler's method

$$0 = \frac{y_{n+1} - y_n}{h} - f(x_n, y_n)$$

on subtracting this from (4), we deduce that

$$e_{n+1} = e_n + h[f(x_n, y(x_n)) - f(x_n, y_n)] + hT_n. \quad \text{3 points}$$

from Lipschitz condition we get

$$|e_{n+1}| \leq (1 + hL)|e_n| + h|T_n|, \quad n = 0, \cdots, N-1$$

By induction, and noting that $1 + hL \leq e^{hL}$

$$\begin{aligned}
|e_n| &\leq \frac{|T_n|}{L}[(1+hL)^n - 1] + (1+hL)^n|e_0| \\
&\leq \frac{|T_n|}{L}\left(e^{L(x_n - x_0)} - 1\right) + e^{L(x_n - x_0)}|e_0|, \\
&\leq \frac{5h}{2L}\left(e^{L(x_n - x_0)} - 1\right) + e^{L(x_n - x_0)}|e_0|, \quad n = 1, \cdots, N
\end{aligned}$$

for $e_0 = y(0) - y_0 = 0, L = 2, x_n = 2$ and $x_0 = 0$, we conclude

$$|e_n| \leq \frac{5h}{4}\left(e^4 - 1\right) = Ch, \quad n = 1, \cdots, N \quad \text{3 points}$$

The Implicit Euler method for (**??**) is given by

$$y_0 = y(x_0)$$
$$y_{n+1} - y_n = hf_{n+1} \quad 0 \le n \le N-1 \quad \text{2 points} \tag{6}$$

Let $y(x)$ be the exact solution of (**??**). We define the Truncation Error (TE) by

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - f(x_{n+1}, y(x_{n+1})) \tag{7}$$

Using Taylor expansion of $y(x_n) = y(x_{n+1} - h)$ around x_{n+1}, we have

$$y(x_n) = y(x_{n+1} - h) \approx y(x_{n+1}) - hy'(x_{n+1}) + \frac{h^2}{2}y''(\psi_{n+1}), \qquad \psi_{n+1} \in (x_n, x_{n+1}) \quad \text{2 points}$$

Substitute in the TE expression gives

$$\begin{aligned} T_n &= \frac{y(x_{n+1}) - y(x_{n+1}) + hy'(x_{n+1}) - \frac{h^2}{2}y''(\psi_{n+1})}{h} - f(x_{n+1} - y(x_{n+1})) \\ &= y'(x_{n+1}) - \frac{h}{2}y''(\psi_{n+1}) - y'(x_{n+1}) \end{aligned}$$

Then

$$|T_n| = \frac{h}{2}|y''(\psi_{n+1})| \le h\frac{M}{2} \quad \text{2 points}$$

Problem 2(16 points) : Given that δ is a positive real number, consider the linear two-step method

$$y_{n+2} - \delta y_{n+1} = \frac{h}{2}(3f_{n+1} - f_n)$$

on the mesh $\{x_n : x_n = x_0 + nh,\ n = 0, \cdots, N\}$ of spacing $h, h > 0$.
a) For which values of δ the method is zero-stable?
b) Is the method convergent for $\delta = 1$? If **No**, justify your answer. If **yes** do the following:

1. Determine the order of accuracy and the error constant.
2. Give a bound for the truncation error T_n.

Solution:

The method is zero stable if and only if its first characteristic polynomial has zeros inside the closed

unit disc, with any which lie on the unit circle being simple.
The characteristic polynomial is

$$\rho(r) = r^2 - \delta r = r(r - \delta) \quad \text{(1 point)}$$

Hence it has two roots $r = 0, r = \delta$, and then the method will be zero stable iff $|\delta| \leq 1$(1 point). Using the assumption δ is real positive, we conclude that our choice of δ for zero stable to be $\delta \in (0, 1]$ (2 points).

For $\delta = 1$ we have

$$y_{n+2} - y_{n+1} = \frac{h}{2}(3f_{n+1} - f_n)$$

Thus, $\rho(r) = r^2 - r$ and $\sigma(r) = \frac{1}{2}(3r - 1)$. The method is consistent if

$$\rho(1) = 0 \quad \text{and} \quad \rho'(1) = \sigma(1) \neq 0. \quad \text{(2 points)}$$

In this method we have

$$\rho(1) = 1 - 1 = 0 \quad \text{and} \quad \rho'(1) = 2(1) - 1 = 1 \neq 0, \;\; \sigma(1) = \frac{1}{2}(3 - 1) = 1 = \rho'(1).$$

So the method is consistent and zero stable for $\delta = 1$ and therefore is convergent. (2 points)

Again for $\delta = 1$ we have

$$y_{n+2} - y_{n+1} = \frac{h}{2}(3f_{n+1} - f_n)$$

The $\alpha_2 = 1, \alpha_1 = -1, \alpha_0 = 0, \;\; \beta_2 = 0, \beta_1 = \frac{3}{2}$, and $\beta_0 = -\frac{1}{2}$.
This method is of order p iff

$$C_0 = C_1 = \cdots C_p = 0 \;\; \text{and} \;\; C_{p+1} \neq 0$$

. where

$$C_0 = \sum_{j=0}^{k} \alpha_j$$
$$C_1 = \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j$$
$$C_2 = \sum_{j=1}^{k} \frac{j^2}{2}\alpha_j - \sum_{j=1}^{k} j\beta_j$$
$$\vdots$$
$$C_q = \sum_{j=1}^{k} \frac{j^q}{q!}\alpha_j - \sum_{j=1}^{k} \frac{j^{q-1}}{(q-1)!}\beta_j$$

for $k = 2$, we have

$$C_0 = 1 - 1 + 0 = 0, \quad \text{(0.5 point)}$$
$$C_1 = (2-1) - (\frac{3}{2} - \frac{1}{2}) = 0, \quad \text{(0.5 point)}$$
$$C_2 = (2 - \frac{1}{2}) - \frac{3}{2} = 0 \quad \text{(0.5 point)}$$
$$C_3 = \frac{8}{6} - \frac{1}{6} - \frac{3}{4} = \frac{5}{12} \neq 0 \quad \text{(0.5 point)}$$

Thus this method is of second order accuracy (2 point) with error constant $C_3 = \frac{5}{12}$.   (1 point)

The truncation error is

$$T_n = \frac{C_3}{\sigma(1)} h^2 y'''(x_n) + O(h^3) \quad \text{(2 points)}$$

where $\sigma(1) = \sum_{j=0}^{2} \beta_j = \frac{3}{2} - \frac{1}{2} = 1 \neq 0$. Thus,

$$T_n = \frac{5}{12} h^2 y'''(x_n) + O(h^3) \quad \text{(1 point)}$$

Problem 3 (12 points): Consider the following two-point BVP:

$$-y''(x) + y'(x) + y(x) = x^2 \quad \text{for } x \in (0,1) \quad \text{with} \quad y(0) = y(1) = 0, \tag{8}$$

a) Develop a second order accurate finite difference scheme for the above BVP.

b) Show (briefly) that the truncation error of the numerical scheme in part a is of order two.

Solution:

We discretise (8) using a finite difference method on the uniform mesh $\{x_i : x_i = ih, \quad i = 0, \cdots, N\}$ of step size $h = 2/N$, $N \geq 2$. Assuming that y is sufficiently smooth, and using Taylor series expansions for the derivatives, we have

$$y''(x_i) = \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} + O(h^2), \tag{9}$$

$$y'(x_i) = \frac{y(x_{i+1}) - y(x_{i-1})}{2h} + O(h^2), \tag{10}$$

Using (9)-(10), we can construct a finite difference method for the numerical solution of (8); we denote by y_i the numerical approximation to $y(x_i)$ for $i = 0, \cdots, N$

$$-\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{y_{i+1} - y_{i-1}}{2h} + y_i = x_i^2 \qquad \text{4 points}$$

After rearranging these, we obtain

$$\left(-\frac{1}{h^2} - \frac{1}{2h}\right) y_{i-1} + \left(\frac{2}{h^2} + 1\right) y_i + \left(-\frac{1}{h^2} + \frac{1}{2h}\right) y_{i+1} = x_i^2, \qquad 1 \leq i \leq N-1$$

This is a system of linear equations of the form

$$A_i y_{i-1} + C_i y_i + B_i y_{i+1} = F_i, \;\; i = 1, \cdots, N-1.$$

The matrix form of the system is

$$M = \begin{bmatrix} C_1 & B_1 & 0 & \cdots & 0 \\ A_2 & C_2 & B_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & B_{N-2} \\ 0 & 0 & 0 & A_{N-1} & C_{N-1} \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{N-1} \end{bmatrix}$$

Substituting the coefficients, we have

$$M = \begin{bmatrix} \left(\frac{2+h^2}{h^2}\right) & \frac{-2+h}{2h^2} & 0 & \cdots & 0 \\ \frac{-2-h}{2h^2} & \left(\frac{2+h^2}{h^2}\right) & \frac{2-h}{2h^2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \frac{2-h}{2h^2} \\ 0 & 0 & 0 & \frac{-2-h}{2h^2} & \left(\frac{2+h^2x}{h^2}\right) \end{bmatrix}$$

$$= \frac{1}{h^2} \begin{bmatrix} (2+h^2) & \frac{h}{2}-1 & 0 & \cdots & 0 \\ -\frac{h}{2}-1 & (2+h^2) & \frac{h}{2}-1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \frac{h}{2}-1 \\ 0 & 0 & 0 & -\frac{h}{2}-1 & (2+h^2) \end{bmatrix}$$

and

$$F = \begin{bmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_{N-1}^2 \end{bmatrix}$$

(b) Using Taylor series expansion, we have

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(x_i) + \cdots \quad \text{1 point} \tag{11}$$

$$y(x_{i-1}) = y(x_i) - hy'(x_i) + \frac{h^2}{2}y''(x_i) - \frac{h^3}{6}y'''(x_i) + \cdots \quad \text{1 point} \tag{12}$$

Adding (11) and (12),

$$y(x_{i+1}) + y(x_{i-1}) = 2y(x_i) + 2h^2y''(x_i) + \frac{h^4}{12}y^{(4)}(x_i) + \cdots$$

$$\Rightarrow y''(x_i) = \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} - \frac{h^2}{12}y^{(4)(x_i)} + \cdots$$

Subtracting (11) and (12),

$$y(x_{i+1}) - y(x_{i-1}) = 2hy'(x_i) + 2\frac{h^3}{6}y'''(x_i) + \cdots$$

$$\Rightarrow y'(x_i) = \frac{y(x_{i+1}) - y(x_{i-1})}{2h} + \frac{h^2}{6}y'''(x_i) + \cdots$$

Then,

$$-y''(x_i) + y'(x_i) + y(x_i) = \left[-\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} + \frac{h^2}{12} y^{(4)(x_i)} + \cdots \right]$$
$$+ \left[\frac{y(x_{i+1}) - y(x_{i-1})}{2h} + \frac{h^2}{6} y'''(x_i) + \cdots \right] + y(x_i) \quad \text{3 points}$$

Therefore, the truncation error

$$T_n = h^2 \left[\frac{1}{12} y^{(4)} + \frac{1}{6} y'''(x_i) \right] \quad \text{3 points}$$

i.e. $|T_n| \leq Ch^2$

Problem 4(10 points) : Give an example of a consistent $O(h^3)$ accurate three-stage RK method (Justify your answer).

Solution: Heun's Method defined by

$$y_{n+1} = y_n + \frac{1}{4} h(k_1 + 3k_3) \quad \text{1 point}$$

where

$$k_1 = f(x_n, y_n),$$
$$k_2 = f(x_n, +\frac{1}{3}h, y_n + \frac{1}{3}hk_1) \quad \text{1 point}$$
$$k_3 = f(x_n + \frac{2}{3}h, y_n + \frac{2}{3}hk_2) \quad \text{1 point}$$

For general explicit one step method

$$y_{n+1} = y_n + h\Phi(x_n, y_n; h)$$

the method is consistent if

$$\Phi(x, y; 0) \equiv f(x, y)$$

For Heun's Method

$$\Phi(x, y; 0) = \frac{1}{4}(k_1 + 3k_3) = \frac{1}{4}(f(x, y) + 3f(x, y)) = f(x, y) \quad \text{2 points}$$

and the method is consistent.
To show that the method is $O(h^3)$ accurate, we estimate the truncation error

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \Phi(x_n, y(x_n); h).$$

using Taylor expansion of $y(x_{n+1})$, we have

$$\begin{aligned} T_n =& y'(x_n) + \frac{h}{2}y''(x_n) + \frac{h^2}{6}y'''(x_n) + O(h^3) \\ & - \left[f + h\frac{1}{2}F_1 + \frac{1}{2}h^2\left(\frac{1}{3}F_1 f_y + \frac{1}{3}F_2\right)\right] \qquad \text{3 points} \end{aligned}$$

where $F_1 = f_x + ff_y$ and $F_2 = f_{xx} + 2ff_{xy} + f^2 f_{yy}$. Using

$$\begin{aligned} &F_1 = y'' \Rightarrow y'' = F_1 \qquad \text{1 point} \\ &F_2 = y''' - y''f_y = y''' - F_1 f_y \Rightarrow y''' = F_2 + F_1 f_y \qquad \text{1 point} \end{aligned}$$

We conclude $T_n = O(h^3)$

Similar argument can be used considering the standard third-order RKM

$$y_{n+1} = y_n + \frac{1}{6}h(k_1 + 4k_2 + k_3)$$

where

$$\begin{aligned} k_1 &= f(x_n, y_n), \\ k_2 &= f(x_n, +\frac{1}{2}h, y_n + \frac{1}{2}hk_1) \\ k_3 &= f(x_n + h, y_n - hk_1 + 2hk_2). \end{aligned}$$

Problem 5(12 points) :: A predictor P and a corrector C are defined by their characteristic polynomials:

$$\begin{aligned} P:\ \rho^*(z) = z^2 - z, \quad \sigma^*(z) = \frac{1}{2}(3z - 1) \\ C:\ \rho(z) = z^2 - z, \quad \sigma(z) = \frac{1}{2}(z^2 + z) \end{aligned}$$

a) Find the stability polynomial $\pi_{P(EC)^mE}$ of this method.

b) Assuming that $m = 1$, use Schur's criterion to calculate the associated intervals of absolute stability. Is this method $A-$ stable? (Justify your answer).

Solution:

The stability polynomial for the $P(EC)^mE$ predictor corrector method is given by

$$\pi_{P(EC)^mE} = \rho(z) - \bar{h}\sigma(z) + M_m(\bar{h})(\rho^*(z) - \bar{h}\sigma^*(z)).$$

where, for $m = 1$,

$$M_m(\bar{h}) = \bar{h}\beta_k = \frac{1}{2}\bar{h}$$

Hence,

$$\pi_{P(EC)^mE}(z) = z^2 - (1 + \bar{h} + \frac{3}{4}\bar{h}^2)z + \frac{\bar{h}^2}{4}. \quad \text{(2 points)} \quad \Rightarrow \quad \pi_{P(EC)^mE}(0) = \frac{\bar{h}^2}{4}$$

Define

$$\hat{\pi}_{P(EC)^mE}(z) = \frac{\bar{h}^2}{4}z^2 - (1 + \bar{h} + \frac{3}{4}\bar{h}^2)z + 1, \quad \text{(2 points)} \quad \Rightarrow \quad \hat{\pi}_{P(EC)^mE}(0) = 1$$

then $|\hat{\pi}_{P(EC)^mE}(0)| > |\pi_{P(EC)^mE}(0)| \iff -2 < \bar{h} < 2$ (2 points) and the polynomial

$$\pi_1(r) = \frac{1}{r}\left[\left(r^2 - (1 + \bar{h} + \frac{3}{4}\bar{h}^2)r + \frac{\bar{h}^2}{4}\right) - \frac{\bar{h}^2}{4}\left(\frac{\bar{h}^2}{4}r^2 - (1 + \bar{h} + \frac{3}{4}\bar{h}^2)r + 1\right)\right] \quad \text{(2 points)}$$

is a Schour polynomial (i.e. its roots r_s satisfies $|r_s| < 1, s = 1, \cdots, k$) iff $\bar{h} \in (-2, 0)$ (2 points). Thus, we conclude that the region of absolute stability is $(-2, 0)$ which is not the whole half plan and therefore the method is not $A-$ stable (2 points).

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Good luck