CE - MATH 571-232

Solution

P1) linear multistep method is said to be A-stable if its region of absolute stability contains the whole of the left half plane.

region of absolute stability

we have $y_{n+1} - y_n = h[\theta f_{n+1} + (1-\theta) f_n]$

using RH critiron to locate the interval of absolute stability.

$\rho(r) = r-1 \qquad \sigma(r) = \theta r + (1-\theta)$

$$\Rightarrow \pi(r;\bar{h}) = \rho(r) - \bar{h}\sigma(r)$$

$$= r - 1 - \bar{h}(r\theta + (1-\theta))$$

$$= (1-\bar{h}\theta) - (1+\bar{h}(1-\theta))$$

apply the transformation

$$r = \frac{1+z}{1-z}$$

$$(1-\bar{h}\theta)\frac{1+z}{1-z} - (1+\bar{h}(1-\theta)) = (1-\bar{h}\theta)(1+z) - (1+\bar{h}(1-\theta))(1-z)$$

$$= \underbrace{(2-2\bar{h}\theta+\bar{h})}_{a_0} z \underbrace{-\bar{h}}_{a_1}$$

$\Rightarrow$ For a_1 to be >0 we must have $\bar{h}<0$

For a_0 to be >0 we need

$$\bar{h} < \frac{2}{2\theta-1}$$

$\Rightarrow$ for $\frac{1}{2} \leq \theta \leq 1 \Rightarrow \bar{h} <$ positive number

taking the intersection with $(-\infty, 0)$

the whole have plane will be included

in the region of absolute stability

and the method is A-stable.

(b) $\rho(z) = z^2 - 3z + 2$ $\quad\quad \sigma(z) = -\frac{1}{2}(z^2 + z)$

$= (z-2)(z-1)$ $\quad\quad = -\frac{1}{2}z(z+1)$

$z=2$ $\quad\quad z=1$

lies outside the

unit disk

The method is not zero-stable.

for Consistency:- $\rho(1)=0$ $\quad \rho'(z)= 2z-3$

$\rho'(1) = 2-3=-1 \neq 0$

$\sigma(1) = -\frac{1}{2}(2) = -1 = \rho'(1)$

and the method is consistent.

(c) $y_n - y_{n-1} = h\left(f_n - 3f_{n-1} + 4f_{n-2}\right)$

$\rho(z) = z^2-1$ $\qquad \sigma(z) = z^2-3z+4$

$= (z-1)(z+1)$

$z=1$ $\qquad z=-1$

simple $\qquad$ simple $\quad \Rightarrow$ zero stable

$\rho'(z) = 2z$

$\rho(1)=0$ $\quad \rho'(1)=2$ $\quad \sigma(1) = 1-3+4=2 \Rightarrow$ Consistent

Zero stable + consistent $\Longrightarrow$ Convergent

to find the order of accuracy:-

we have

$\alpha_0 = -1 \quad \alpha_1 = 0 \quad \alpha_2 = 1 \ , \ \beta_0 = 4 \ , \ \beta_1 = -3 \ , \ \beta_2 = 1$

$C_0 = \alpha_0 + \alpha_1 + \alpha_2 = 0.$

$C_1 = 2\alpha_0 + \alpha_1 - (\beta_0 + \beta_1 + \beta_2) = 2 + 0 - 2 = 0$

$C_2 = \frac{4}{2}\alpha_2 + \frac{1}{2}\alpha_1 - (2\beta_2 + \beta_1) = 2 - 2(2-3) = 4 \neq 0$

The method is of order 1 with error constant $C_2 = 4$

P2

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

(a) It is an explicit method. we can deduce it from Butcher tableau, noticing that the $(a_{i,j})_{i,j}$ is such that $a_{ij} = 0$, $i \leq j$.

(b) $k_1 = y_n = f(x_n, y_n)$

$$k_2 = f(x_n + h, y_n + hk_1) = f(x_{n+1}, y_n + h f(x_n, y_n))$$

$$y_{n+1} = y_n + \frac{h}{2}\Big[\underbrace{f(x_n, y_n)}_{k_1} + \underbrace{f(x_{n+1}, y_n + h f(x_n, y_n))}_{k_2}\Big]$$

Check if it is consistent:-

since $\sum_{j=1}^{2} a_{ij} = c_i$, $i = 1, 2$ & $\sum_{j=1}^{n} b_j = 1$

$\Rightarrow$ the method is consistent.

where $c_1 = 0$, $c_2 = 1$, $b_1 = b_2 = \frac{1}{2}$, $a_{11} = a_{12} = a_{22} = 0$, $a_{21} = 1$.

(c) We have

$$\frac{y_{n+1} - y_n}{h} = \frac{1}{2} f(x_n, y_n) + \frac{1}{2} f(x_{n+1}, y_n + h f(x_n, y_n))$$

The Taylor expansion of the last summand (with $y(x_n)$ instead of y_n) is

$$f\Big(x_{n+1}, y(x_n) + h f(x_n, y(x_n))\Big) = f\big(x_{n+1}, y(x_n) + h y'(x_n)\big)$$

$$= f(x_n, y(x_n)) + h \frac{\partial f}{\partial x}(x_n, y(x_n)) + h \frac{\partial f}{\partial y}(x_n, y(x_n))$$
$$+ \frac{h^2}{2}\Big[\frac{\partial^2 f}{\partial x^2}(x_n, y(x_n)) + y'^2(x_n) \frac{\partial^2 f}{\partial y^2}(x_n, y(x_n))$$
$$+ 2 y'(x_n) \frac{\partial^2 f}{\partial x \partial y}(x_n, y(x_n)) \Big] + O(h^3)$$

$$= y'(x_n) + h y''(x_n) + \frac{h^2}{2}\Big[\frac{\partial^2 f}{\partial x^2} + y'^2(x_n) \frac{\partial^2 f}{\partial y^2} + 2 y'(x) \frac{\partial^2 f}{\partial x \partial y}\Big]$$
$$+ O(h^3)$$

Then,

$$\frac{y(x_{n+1}) - y(x_n)}{h} - \frac{1}{2} f(x_n, y(x_n)) - \frac{1}{2} f\Big(x_{n+1}, y(x_n) + h f(x_n, y(x_n))\Big)$$

$$= y'(x_n) + \frac{1}{2} h y''(x_n) + \frac{1}{6} h^2 y'''(x_n) + O(h^3) - \frac{1}{2} y'(x_n) - \frac{1}{2} y'(x_n)$$
$$- \frac{1}{2} y''(x_n) - \frac{h^2}{4}\Big[\frac{\partial^2 f}{\partial x^2}(x_n, y(x_n)) + (y'(x_n))^2 \frac{\partial^2 f}{\partial^2 y}(x_n, y(x_n))$$
$$+ 2 y'(x_n) \frac{\partial^2 f}{\partial x \partial y}(x_n, y(x_n))\Big] + O(h^3)$$
$$= h^2\Big[\Big(\frac{1}{6} y'''(x_n) - \frac{1}{4} \frac{\partial^2 f}{\partial x^2}(x_n, y(x_n)) - \frac{1}{4}(y'(x_n)) \frac{\partial^2 f}{\partial y^2} - \frac{1}{2} u'(x_n) f(x_n, y(x_n))\Big]$$
$$+ O(h^3)$$

$= O(h^2) \Rightarrow$ The method is second order.

P3 using Taylor series the truncation error

$$T_n = \frac{1}{h\sigma(1)}\left[C_0\, y(x_n) + C_1 h\, y'(x_n) + C_2 h^2 y''(x_n) + \cdots\right]$$

$$y_{n+1} - (1-a)\, y_n - a\, y_{n-1} = \frac{h}{12}\left\{(5-a) f_{n+1} + 8(1+a) f_n + (5a-1) f_{n-1}\right\}$$

$$\sigma(z) = z^2 - (1-a) z - a$$

$$\rho(z) = \frac{5-a}{12} z^2 + \frac{8(1+a)}{12} z + \frac{5(a-1)}{12}$$

$\alpha_2 = 1 \quad \alpha_1 = a-1 \quad \alpha_0 = -a$

$\beta_2 = \frac{5-a}{12} \quad \beta_1 = \frac{8(1+a)}{12} \quad \beta_0 = \frac{5a-1}{12}$

as in problem 1 compute

$C_0 = 0,\ C_1 = 0,\ C_2 = 0,\ C_3 = 0,$

until reach $C_4 = \frac{a}{24} - \frac{1}{24} \neq 0$ (for $a \neq 1$)

and the method is $O(h^3)$

$$C_5 = \frac{31}{5!} + \frac{a}{5!} - \frac{1}{36}(9-a) \neq 0$$

Then for $a=1$, $C_4=0$ and $C_5 \neq 0 \Rightarrow$ the method is $O(h^4)$

for $a=1$

$$y_{n+2} - y_n = \frac{h}{3}\left[f_{n+2} + 4 f_{n+1} + f_n \right]$$

$$\sigma(z) = z^2 - 1 = (z-1)(z+1)$$

the roots $z=1$ and $z=-1$ are simple root in the unit disk $\Rightarrow$ the scheme satisfies the root condition

(c) Similar to problem (1).

P4 using FDM on the uniform mesh

$$\{x_i : x_i = ih,\ i = 0, \dots, N\}$$

of step size $h = 2/N$, $N \geq 2$

Assuming that y is sufficiently smooth, and using Taylor series expansions for the derivatives, we have

$$y''(x_i) = \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} + O(h^2) \rightarrow ①$$

$$y'(x_i) = \frac{y(x_{i+1}) - y(x_{i-1})}{2h} + O(h^2) \rightarrow ②$$

using ① and ② we can construct a finite difference method for the numerical approximation to $y(x_i)$ for $i = 1, \dots, N$

$$-\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{y_{i+1} - y_{i-1}}{2h} + y_i = x_i^2$$

After rearranging these, we obtain

$$\left(-\frac{1}{h^2} - \frac{1}{2h}\right) y_{i-1} + \left(\frac{2}{h^2} + 1\right) y_i + \left(-\frac{1}{h^2} + \frac{1}{2h}\right) y_{i+1} = x_i^2$$

$$1 \leq i \leq N-1$$

This is a system of linear equations of the form

$$A_i y_{i-1} + C_i y_i + B_i y_{i+1} = F_i, \quad i = 1, \dots, N-1$$

The matrix form of the system is

$$M = \begin{bmatrix} C_1 & B_1 & 0 & \cdots & 0 \\ A_2 & C_2 & B_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & B_{N-2} \\ 0 & 0 & & A_{N-1} & C_{N-1} \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{bmatrix}$$

Substituting the coefficients, we have

$$M = \frac{1}{h^2} \begin{bmatrix} (2+h^2) & \frac{h}{2}-1 & 0 & - - - & 0 \\ -\frac{h}{2}-1 & 2+h^2 & \frac{h}{2}-1 & - - - - & 0 \\ \cdots & - & - & - - - & \frac{h}{2}-1 \\ - & - & - - - & -\frac{h}{2}-1 & 2+h^2 \end{bmatrix}$$

$$F = \begin{bmatrix} x_1^2 & x_2^2 & - - - & x_{N-1}^2 \end{bmatrix}^T$$

and M is tridiagonal matrix.

(b) Using Taylor Series expansion, we have

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2} y''(x_i) + \frac{h^3}{6} y'''(x_i) + \cdots \quad (1)$$

$$y(x_{i-1}) = y(x_i) - hy'(x_i) + \frac{h^2}{2} y''(x_i) - \frac{h^3}{6} y'''(x_i) + \cdots \quad (2)$$

$$(1)+(2) \Rightarrow y(x_{i+1}) + y(x_{i-1}) = 2y(x_i) + 2h^2 y''(x_i) + \frac{h^4}{12} y^{(4)}(x_i) + \cdots$$

$$\Rightarrow y''(x_i) = \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} - \frac{h^2}{12} y^{(4)}(x_i) + \cdots$$

$$(1)-(2) \Rightarrow y(x_{i+1}) - y(x_{i-1}) = 2hy'(x_i) + 2\frac{h^3}{6} y'''(x_i) + \cdots$$

$$\Rightarrow y'(x_i) = \frac{y(x_{i+1}) - y(x_{i-1})}{2h} + \frac{h^2}{6} y'''(x_i) + \cdots$$

Then,

$$-y''(x_i) + y'(x_i) + y(x_i) = \left[- \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} + \frac{h^2}{12} y^{(4)}(x_i) + \cdots \right]$$
$$+ \frac{y(x_{i+1}) - y(x_{i-1})}{2h} + \frac{h^2}{6} y'''(x_i) + \cdots \Big] + y(x_i)$$

$$\Rightarrow T_n = h^2 \left[\frac{1}{12} y^{(4)} + \frac{1}{6} y'''(x_i) \right]$$

$$|T_n| \le Ch^2$$

P6 $y' = \ln(\ln(4+y^2))$, $x \in [0,1]$, $y(0)=1$

(a) The explicit Euler method

$$y_0 = y(0)$$

$$(*) \quad y_{n+1} = y_n + h\, f(x_n, y_n) = y_n + h \ln(\ln(4+y^2))$$

$$n = 0, 1, \dots, N.$$

Truncation error similar to problem 2

(b) Recall that

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - f(x_n, y(x_n)) \longrightarrow (1)$$

and from $(*)$ we have

$$0 = \frac{y_{n+1} - y_n}{h} - f(x_n, y_n) \longrightarrow (2)$$

Subtracting (2) from (1)

$$e_{n+1} = e_n + h\left[f(x_n, y(x_n)) - f(x_n, y_n) \right] + h T_n$$

assuming f satisfies Lipschitz condition

$$|f(x_n, y(x_n)) - f(x_n, y_n)| < L\, |y(x_n) - y_n|$$

$\Rightarrow$

$$|e_{n+1}| \leq (1 + hL)|e_n| + h\,|T_n| \quad , \quad n = 0, \ldots, N-1$$

(c)

$$|e_{n+1}| \leq (1+hL)\,|e_n| + h^2/4 \quad , \quad n = 0, \ldots, N-1$$

By induction and noting that $1+hL \leq e^{hL}$

$$|e_n| \leq \frac{h^2}{4}\left[\frac{(1+hL)^n - 1}{Lh} \right] + (1+hL)^n\, |e_0|$$

$$\leq \frac{h}{4L}\left[e^{L(x_n - x_0)} - 1 \right] + (1+hL)^n\, |e_0|$$

for $x_0 = 0$ & $e_0 = 0$ and $L = \frac{1}{2\ln 4}$

$$|e_n| \leq \frac{2\ln 4\, h}{4}\left(e^{\frac{x_n}{\ln 8}} - 1\right), \quad n = 0, \dots, N-1$$

$$|e_n| < h\,\frac{\ln 4}{2}\left(e^{\frac{1}{\ln 8}} - 1\right)$$

$$< h\left(\frac{e}{2} - \frac{1}{2}\right)$$

$$|e_n| \leq 10^{-4} \Leftarrow h < \frac{2}{e-1} \times 10^{-4}$$

$$h = \frac{b-a}{N} = \frac{1}{N}$$

we want

$$\Rightarrow N > \frac{2}{e-1} \times 10^{-4} \Rightarrow N_0 = 1$$

For problem 5 See Page 50 in Endre Süli book.