Problem 1 (20 points) ((a): 10 points ; (b): 10 points)

Consider the function

$$f(x) = x^3 + 1$$

(a) We have $x_0 = 0$, $x_1 = \frac{1}{2}$ and $x_2 = 1$.

The degree 2 Lagrange interpolant is given by

$$P_2(x) = f(0)\, l_0(x) + f\left(\tfrac{1}{2}\right) l_1(x) + f(1)\, l_2(x)$$

where

$$l_0(x) = \frac{\left(x-\frac{1}{2}\right)(x-1)}{\left(0-\frac{1}{2}\right)(0-1)} = 2\left(x-\frac{1}{2}\right)(x-1)$$

$$l_1(x) = \frac{(x-0)(x-1)}{\left(\frac{1}{2}-0\right)\left(\frac{1}{2}-1\right)} = -4x(x-1)$$

$$l_2(x) = \frac{(x-0)\left(x-\frac{1}{2}\right)}{(1-0)\left(1-\frac{1}{2}\right)} = 2x\left(x-\frac{1}{2}\right)$$

Thus

$$P_2\left(\tfrac{1}{4}\right) = l_0\left(\tfrac{1}{4}\right) + \frac{9}{8}\, l_1\left(\tfrac{1}{4}\right) + 2\, l_2\left(\tfrac{1}{4}\right) = \frac{31}{32}.$$

The absolute Error is given by

$$|\text{error}| = \left| P_2\left(\tfrac{1}{4}\right) - f\left(\tfrac{1}{4}\right)\right| = \left|\frac{31}{32} - \frac{65}{64}\right| = \frac{3}{64}$$

(b) We have

$$|\text{error}| = \left| \left(\tfrac{1}{4}-x_0\right)\left(\tfrac{1}{4}-x_1\right)\left(\tfrac{1}{4}-x_2\right) \frac{f^{(3)}(\xi)}{3!} \right|$$

where $\xi \in [0,1]$. But since $f^{(3)}(x) = 6$.

We obtain $|\text{error}| = \left| \left(\tfrac{1}{4}-0\right)\left(\tfrac{1}{4}-\tfrac{1}{2}\right)\left(\tfrac{1}{4}-1\right) \right| = \frac{3}{64}$

Problem 2 (25 points) (a): 12 points ; (b): 13 points

(a) We shall show by induction that

$$f[x, x+h, \dots, x+nh] = \frac{1}{n!\,h^n} \Delta_h^n f(x)$$

- for $n=0$, This is trivial
- Assuming validity to order n, we have

$$f[x, x+h, \dots, x+nh, x+(n+1)h]$$

$$= \frac{f[x+h, \dots, x+(n+1)h] - f[x, x+h, \dots, x+nh]}{nh}$$

$$= \frac{\frac{1}{n!\,h^n} \Delta_h^n f(x+h) - \frac{1}{n!\,h^n} \Delta_h^n f(x)}{nh}$$

$$= \frac{1}{(n+1)!\,h^{n+1}} \Delta_h^{n+1} f(x).$$

(b) From the Newton's divided diffence interpolation formula, we have

$$P_n(x) = f[x_0] + \sum_{j=1}^{n} f[x_0, x_0+h, \dots, x_0+jh] \prod_{i=0}^{j-1} (x-(x_0+ih))$$

$$= \Delta_h^0 f(x_0) + \sum_{j=1}^{n} \frac{1}{j!\, h^j} \Delta_h^j f(x_0) \prod_{i=0}^{j-1} (x-(x_0+ih))$$

Now

$$\frac{\prod_{i=0}^{j-1} (x-(x_0+ih))}{j!\, h^j} = \frac{\prod_{i=0}^{j-1}(s-i)}{j!} = \binom{s}{j}; \quad s = \frac{x-x_0}{h}$$

Thus

$$P_n(x) = \sum_{j=0}^{n} \binom{s}{j} \Delta_h^j f(x_0)$$

Problem 3. (15 points) (a): 6pts, (b): 6pts, (c): 3pts

Consider the function

$$f(x) = a^x \quad \text{with} \quad a>1 \quad \text{a real number}$$

(a) We shall show by induction that.

$$\Delta_1^k f(x) = (a-1)^k a^x$$

For $k=0$, it is trivial

assuming the Validity for k, we have

$$\Delta_1^{k+1} f(x) = \Delta_1^k f(x+1) - \Delta_1^k f(x)$$

$$= (a-1)^k a^{x+1} - (a-1)^k a^x$$

$$= (a-1)^k a^x (a-1) = (a-1)^{k+1} a^x.$$

(b) According to problem 2, with $h = 1$ & $x_0 = 0$

we have

$$P_n(x) = \sum_{k=0}^{n} \binom{x}{k} \Delta_1^k f(x_0)$$

$$= \sum_{k=0}^{n} \binom{x}{k} (a-1)^k$$

(c) Taking $a = 1, 2, 3, \ldots, n+1$, we have. for degree 3. interpolant polynomial

$$\sum_{k=0}^{3} \binom{3}{k} = 2^3$$

$$\sum_{k=0}^{3} 2^k \binom{3}{k} = 3^3$$

$$\vdots$$

$$\sum_{k=0}^{3} n^k \binom{3}{k} = (n+1)^3$$

Therefor

$$\sum_{k=0}^{3}\left(1^k+2^k+\dots+n^k\right)\binom{3}{k} = 2^3+3^3+\dots+(n+1)^3.$$

Therefor

$$n+\frac{3n(n+1)}{2}+3\left(1^2+2^2+\dots+n^2\right)+\left(1^3+2^3+\dots+n^3\right)$$
$$= 2^3+3^3+\dots+(n+1)^3$$

Thus

$$n+\frac{3n(n+1)}{2}+3\left(1^2+2^2+\dots+n^2\right) = (n+1)^3-1.$$

and computation gives.

$$1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

$\longleftrightarrow$

Problem 4 (15 points). (a): 8 pts (b): 7 pts.

(a) We have

$$L_j(x) = \prod_{\substack{k=1\\k\neq j}}^{n}\frac{x-x_k}{x_j-x_k} = A_j\prod_{\substack{k=1\\k\neq j}}^{n}(x-x_k) \qquad A_j\text{: Constant}$$

Therefor

$$L_j(x)\,L_i(x) = A_jA_i\,\Pi_n(x)\,P_{n-2}(x) \qquad i\neq j$$

with $P_{n-2}(x)$ is a polynomial of degree $n-2$.

Thus
$$\int_a^b L_j(x)L_i(x)w(x)\,dx = \int_a^b A_jA_i\,\Pi_n(x)\,P_{n-2}(x)\,w(x)\,dx = 0.$$

(b) We have

$$P_{n-1}(x) = \sum_{k=1}^{n} y_k \, l_k(x)$$

using orthogonality of the Lagrange polynomials we obtain

$$\|P_{n-1}\|^2 = \sum_{k=1}^{n} y_k^2 \, \|L_k\|^2$$

Problem 5 (15 points) (a): 5 pts ; (b): 5 pts ; (c): 5 pts.

(a) Integration by part gives

$$\int_a^b S''(x)\, D''(x)\, dx = S''(x)\, D'(x)\Big|_a^b - \int_a^b S^{(3)}(x)\, D'(x)\, dx$$

$$= - \int_a^b S^{(3)}(x)\, D'(x)\, dx$$

Since $D'(a) = 0$ and $D'(b) = 0$.

(b) We have

$$\int_a^b S^{(3)}(x)\, D'(x)\, dx = \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} S^{(3)}(x)\, D'(x)\, dx$$

$$= \sum_{i=1}^{n-1} \Big(\underbrace{S^{(3)}(x)\, D(x)\Big|_{x_i}^{x_{i+1}}}_{=0} - \underbrace{\int_{x_i}^{x_{i+1}} S^{(4)}(x)\, D(x)\, dx}_{=0} \Big) = 0.$$

(c) We have

$$f(x) = D(x) + S(x) \Rightarrow f''(x) = D''(x) + S''(x)$$

Thus

$$\int_a^b (f''(x))^2\,dx = \int_a^b (D''(x))^2\,dx + 2\underbrace{\int_a^b D''(x)\,S''(x)\,dx}_{=0}$$

$$+ \int_a^b (S''(x))^2\,dx$$

$$\geq \int_a^b (S''(x))^2\,dx$$

Problem 6 (10 points)

Let Q be an arbitrary polynomial of degree m

$$Q(x) = \sum_{i=0}^{m} d_i\,\Pi_i(x) \qquad Q \neq Q^*$$

Then

$$\|Q - P\|^2 = \sum_{i=0}^{m} (d_i - a_i)^2 + \sum_{i=m+1}^{n} a_i^2 >$$

$$> \sum_{i=m+1}^{n} a_i^2 = \|Q^* - P\|^2$$