Department of Mathematics and Statistics, KFUPM
Math-580-Comprehensive, Year 2020-2021

MM: 100 Duration: 120 minutes

Q1 (5 + 5 pts) : **(a) Let Ω_1 and Ω_2 be convex subsets of $\mathbb{R}^n$. Prove that $\Omega_1 + \Omega_2$ is a convex set of $\mathbb{R}^n$.**
(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex function and let $\psi : \mathbb{R}^n \to \bar{\mathbb{R}}$ be non-decreasing and convex on a convex set containing the range of the function f. Show that $\psi o f$ is convex.

Q2 (12pts) : **Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function on a nonempty convex set Ω in $\mathbb{R}^n$. Prove that the following are equivalent.**
(i) f is convex.
(ii) $f(x) - f(\bar{x}) \geq \nabla f(\bar{x})(x - \bar{x})$ **for all** $x, \bar{x} \in \Omega$

Q3 (8pts) : **Show that the following problem is a convex optimization problem**

$$\textbf{Minimize } f(x_1, x_2) = \sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2 + (x_3 - x_6)^2}$$

$$\textbf{subject to} \quad (x_4 - 3)^2 + x_5^2 \leq 1,$$

$$4 \leq x_6 \leq 7.$$

Q4 (1 + 4 + 6 pts) : **(a) Define a normal cone to a convex set at a point.**

(b) Let $\bar{x} \in \Omega$ for a convex subset Ω of $\mathbb{R}^n$. Then prove that $N(\bar{x}; \Omega)$ is a closed, convex cone containing the origin.

(c) Define domain and epigraph of an extended real-valued function f. Show that $\partial f(\bar{x}) = N((\bar{x}; dom f)\}$, where $\partial f(\bar{x})$ is the singular subdifferential of f at $\bar{x} \in dom f$.

Q5 (5 + 5 + 4 pts) : **(a) Determine the subdifferential of the convex function**

$$f(x) = \begin{cases} -x, & if \quad x < 0 \\ x^2 & if \quad x \geq 0 \end{cases}$$

at the points -1 and 0.

(b) Use the the definition of directional derivative to determine $\partial f(0)$, where

$$f(x_1, x_2) = |x_1| + x_2^2.$$

(c) Let $f_1(x) = -x$ **and** $f_2(x) = x, x \in R.$ **use the result**

$$\partial(max f_i)(\bar{x}) \supset co\{\cap_{i \in I(\bar{x})} \partial f_i(\bar{x})\},, i = 1, 2, \cdots, m$$

to compute $\partial f(0)$.

Or

(c) Define Gateaux derivative of an extended real-valued function at the fixed point. If f **is a Gateaux differentiable at** $\bar{x}$**, then show that** $\partial f(\bar{x})$ **is a singleton.**

Q6 (**10pts**) : **Compute the Fenchel conjugate of**

$$f(x) = \begin{cases} \dfrac{x^p}{p}, & if \quad x \geq 0 \\ \\ \infty & if \quad x < 0, \end{cases}$$

where $x \in R, p \in R, p > 1.$

Q7 ($\mathbf{3 + 7}$ **pts**) : **(a) Define polar cone, dual cone and tangent cone to a convex set at a point.**

(b) Let Ω **be a non-empty convex set in** R^n**. Let** $\bar{x} \in \Omega$**. Then prove that the normal cone of** Ω **at** $\bar{x}$ **is the polar cone of the tangent cone of** Ω **at** $\bar{x}$ **(That is** $N(\bar{x}; \Omega) = (T(\bar{x}; \Omega))^*$**).**

Or

(b) Let Ω **be a non-empty convex set in** R^n**. Let** Ω_1 **and** Ω_2 **be convex sets with** $int(\Omega_1 \bigcap \Omega_2) \neq \phi$**. Show that**

$$T(\bar{x}; \Omega_1 \bigcap \Omega_2) = T(\bar{x}; \Omega_1) \bigcap T(\bar{x}; \Omega_2)$$

for any $\bar{x} \in \Omega_1 \bigcap \Omega_2$.

Q8 (**15 pts**) : **Use Karush-Kuhn-Tucker necessary conditions to determine all solutions of the following problem:**

Maximize $f(x_1, x_2) = (x_1 + 2)^2 + (x_2 - 1)^2$

subject to $\quad -x_1 + x_2 - 2 \leq 0$

$$x_1^2 - x_2 \leq 0.$$

Q9 (**10pts**) : **Sketch the feasible set defined by**

$$S = \{(x_1, x_2) : x_2 - 2 \leq 0, \; 1 + (x_1 - 1)^2 - x_2 \leq 0, \}.$$

Find the set of the feasible direction at point $(1, 1)$ **of the feasible set** S**.**

(5+5) ①

a) ~~Define a convex set~~. Let $\Omega_1, \Omega_2 \subset R^n$ be convex sets ~~and let $\lambda \in R$~~. Then prove that $\Omega_1 + \Omega_2$ is also convex set in R^n.

Sol: ~~Definition~~

Fix any $x, y \in \Omega_1 + \Omega_2$ and $\lambda \in (0,1)$. Then

$x = x_1 + x_2$; $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$

$y = y_1 + y_2$; $y_1 \in \Omega_1$ and $y_2 \in \Omega_2$

Thus

$$\lambda x + (1-\lambda) y = \lambda (x_1 + x_2) + (1-\lambda)(y_1 + y_2)$$
$$= \underbrace{\lambda x_1 + (1-\lambda) y_1}_{\in \Omega_1} + \underbrace{\lambda x_2 + (1-\lambda) y_2}_{\in \Omega_2}$$
$$\in \Omega_1 + \Omega_2$$

Therefore $\Omega_1 + \Omega_2$ is a convex set.

b) ~~Define a convex function on a convex set~~. Let $f : R^n \to R$ be a convex function and let $\phi : R \to \bar{R}$ be nondecreasing and convex on a convex set containing the range of the function f. Show that $\phi \circ f$ is convex.

Proof :- ~~Def~~

Take any $x_1, x_2 \in R^n$ and $\lambda \in (0,1)$, Then we have by the convexity of f that

$$f(\lambda x_1 + (1-\lambda) x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

Since ϕ is nondecreasing and it is also convex,

$$(\phi \circ f)(\lambda x_1 + (1-\lambda) x_2) = \phi\left(f(\lambda x_1 + (1-\lambda) x_2)\right)$$
$$\le \phi(\lambda f(x_1) + (1-\lambda) f(x_2))$$
$$\le \lambda \phi(f(x_1)) + (1-\lambda) \phi(f(x_2))$$
$$= \lambda (\phi \circ f)(x_1) + (1-\lambda)(\phi \circ f)(x_2)$$

| lhs = a, Rhs b
| ϕ is non d.
| $a \le b$
| $\phi(a) \le \phi(b)$

Q. 1 (a) 2

Let $f : R^n \to R$ be differentiable over an open domain. Prove that following are equivalent

(i) f is convex

(ii) $f(y) \geq f(x) + \nabla f(x)^T (y-x)$, for all $x, y \in dom(f)$

or $y \to \bar{x}$, $x \to \bar{x}$

Sol: (i) ⇒ (ii)

$$f(\lambda y + (1-\lambda)x) \leq \lambda f(y) + (1-\lambda) f(x),$$

$$f(x + \lambda(y-x)) \leq f(x) + \lambda (f(y) - f(x))$$

$$f(y) - f(x) \geq \frac{f(x + \lambda(y-x)) - f(x)}{\lambda} \quad \forall \lambda \in (0,1]$$

As $\lambda \to 0$, we get

$$f(y) - f(x) \geq \nabla f(x)^T (y-x) \quad \text{— (1)}$$

(by Taylor series)

(ii) ⇒ (i) Suppose (1) holds $\forall x, y \in dom(f)$, Take any $x, y \in dom(f)$ and let

$$Z = \lambda x + (1-\lambda) y$$

We have

$$f(x) \geq f(z) + \nabla f(z)^T (x-z) \quad \times \lambda$$

$$f(y) \geq f(z) + \nabla f(z)^T (y-z) \quad \times 1-\lambda$$

Adding

$$\lambda f(x) + (1-\lambda) f(y) \geq f(z) + \nabla f(z)^T (\lambda x + (1-\lambda) y - z)$$

$$= f(z)$$

$z - z = 0$

$$= f(\lambda x + (1-\lambda) y)$$

$$f(x) = f(\bar{x}) + f'(x)(x-\bar{x}) + \frac{f''(x)}{2}(x-\bar{x})^2 + \cdots$$

$$f(x) = f(\bar{x}) + \nabla f(\bar{x})(x-\bar{x}) + \|x-\bar{x}\| \lambda(\bar{x}; x-\bar{x})$$

where $\lim_{x \to \bar{x}} \lambda(\bar{x}; x-\bar{x}) = 0$

Show that the fallowing problem is a convex optimization problem

$$\text{Minimize} \quad (x_1 - x_4)^2 + (x_2 - x_5)^2 + (x_3 - x_6)^2$$

$$(x_4 - 3)^2 + x_5^2 \leq 1$$

$$4 \leq x_6 \leq 7$$

Sol: objective function.

$$\sqrt{\quad} = \left\| \begin{matrix} x_1 - x_4 \\ x_2 - x_5 \\ x_3 - x_6 \end{matrix} \right\|$$

Norm is a cx function.

Hessian of $g_1(x) = (x_4 - 3)^2 + x_5^2 - 1$ is

$$\nabla^2 g_1(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hessian is PSD and so the function $g_1(x)$ is convex. Consequently set $g_1(x) \leq 0$ is convex.

Set $4 \leq x_6 \leq 7$ is the intersection of hyperplanes and so cx.

$\Rightarrow$ Problem is cx.

Q.4 (1+4+6): Let $\Omega \subset \mathbb{R}^n$ be a convex set with $\bar{x} \in \Omega$;

The normal cone to Ω at $\bar{x}$ is

$$N(\bar{x};\Omega) = \{v \in \mathbb{R}^n ; \langle v, x-\bar{x}\rangle \le 0 \text{ for all } x \in \Omega\}$$

(b) We will show $N(\bar{x},\Omega)$ is a convex cone — [$\lambda x \in \Omega$, $x \in \Omega$, $\lambda \ge 0$]

By the definition

$$\langle v, x-\bar{x}\rangle \le 0 \quad \forall x \in \Omega$$

$$\langle \lambda v, x-\bar{x}\rangle = \lambda \langle v, x-\bar{x}\rangle, \quad \forall x \in \Omega$$

$\lambda \ge 0$, $\langle v, x-\bar{x}\rangle \le 0$

≤ 0

Thus $\lambda v \in N(\bar{x};\Omega)$ $\Rightarrow N(\bar{x};\Omega)$ is a convex cone.

To check its closedness: Fix a sequence $\{v_k\} \subset N(\bar{x};\Omega)$ that converges to v

Then passing to limit in

$$\langle v_k, x-\bar{x}\rangle \le 0 \quad \forall x \in \Omega \text{ as } k \to \infty$$

yields $\langle v, x-\bar{x}\rangle \le 0$ whenever $x \in \Omega$ and so $v \in N(\bar{x};\Omega)$

(c) $f : \mathbb{R}^n \to \bar{\mathbb{R}}$

$$\text{dom } f = \{x \in \mathbb{R}^n : f(x) < \infty\}$$

$$\text{epi} f = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : x \in \mathbb{R}^n, t \ge f(x)\}$$

$$= \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : x \in \text{dom } f, t \ge f(x)\}$$

definition: $\partial f(x) = \{v \in \mathbb{R}^n : (v,0) \in N((\bar{x}; f(\bar{x})); \text{epi} f)\}$

Fix any $v \in \partial f(x)$ and observe that $(x, f(x)) \in \text{epi} f$,

$x \in \text{dom } f$ and

Using def. of $\partial f(x)$ and normal cone give us

$$\langle v, x-\bar{x}\rangle = \langle v, x-\bar{x}\rangle + 0\,(f(x)-f(\bar{x})) \le 0$$

which show that $v \in N(\bar{x}; \text{dom } f)$.

Conversely: Suppose that $v \in N(\bar{x}; \text{dom } f)$ and fix any $(x,\lambda) \in \text{epi} f$. Then we have $f(x) \le \lambda$ and so $x \in \text{dom} f$.

Thus

$$\langle v, x-\bar{x}\rangle + 0\,(\lambda - f(\bar{x})) = \langle v, x-\bar{x}\rangle \le 0$$

which implies that

$$(v,0) \in N((\bar{x}; f(\bar{x})); \text{epi} f)$$

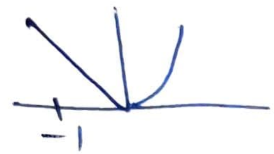
i.e. $v \in \partial f(x)$

5 (a) Define a subgradient of a convex function. Use the definition of subgradient to determine the subdifferential of convex function

$$f(x) = \begin{cases} -x & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

at the points -1 and 0.

Sol. Def. $\langle v, x-\bar{x} \rangle \leq f(x) - f(\bar{x}) \quad \forall x \in R^n$, $\bar{x} \in \text{dom} f$, $v \in R^n$, $f: R^n \to \bar{R}$



At $\bar{x} = -1$:

$f(x) \geq 1 + v(x+1) \quad \forall x \in R$

$\Rightarrow -x \geq 1 + v(x+1), \quad x < 0$

$x^2 \geq 1 + v(x+1) \quad x \geq 0$

Setting for example $x = -2$ and $x = -0.5$ we obtain

$v \geq -1 \quad \& \quad v \leq -1$

Hence $v = -1$ is the unique solution of the system

$2 \geq 1 - v \Rightarrow v \geq -1$ | ~~$4 \geq 1 +$~~ $v +$

At $\bar{x} = 0$:

$f(x) \geq vx \quad \forall x \in R$

or

$-x \geq vx, \quad x < 0$

$x^2 \geq vx, \quad x \leq 0$

$\Leftrightarrow -v \leq +1, \quad x < 0$ or $v \geq -1$

$x \geq v, \quad x > 0$

$\Rightarrow \boxed{-1 \leq v \leq 0}$

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~~Let~~ Define the directional derivative of an extended real function at a fixed point. Use the definition of directional derivative to determine $\partial f(0)$, where

$$f(x_1, x_2) = |x_1| + x_2^2$$

$$f'(\bar{x}; d) = \lim_{t \to 0} \frac{f(\bar{x} + td) - f(\bar{x})}{t}$$

$$v \in \partial f(\bar{x}), \quad f'(\bar{x}; d) \geq \langle v, d \rangle$$

$\bar{x} = (0,0)$, $d = (d_1, d_2)$

$$f'(\bar{x}; d) = \lim_{t \to 0} \frac{|td_1| + (td_2)^2}{t}$$

$d_1 > 0$, $\lim_{t \to 0^+} \frac{td_1 + t^2 d_2^2}{t} = d_1$ | $d_1 < 0$ $\lim_{t \to 0^-} \frac{-td_1 + t^2 d_2}{t} = -d_1$

Now if $d_1 > 0$, $d_2 = 0$

then $f'(0; d) \geq \langle v, d \rangle$

$d_1 \geq v_1 d_1 \quad \Rightarrow v_1 \leq 1$

if $d_1 < 0$, $d_2 = 0$: $-d_1 \geq v_1 d_1 \quad \Rightarrow v_1 \geq -1$

$$\boxed{-1 \leq v \leq 1}$$

If $d_1 = 0$, $d_2 = 1$; $f'(0; d) \geq v_1 d_1 + v_2 d_2$

$0 \geq v_1(0) + v_2 \Rightarrow v_2 \leq 0$

If $d_1 = 0$, $d_2 = -1$; $0 \geq v_1(0) + v_2(-1)$

$0 \geq -v_2 \quad \Rightarrow v_2 \ngeq 0$

$$\boxed{v_2 = 0}$$

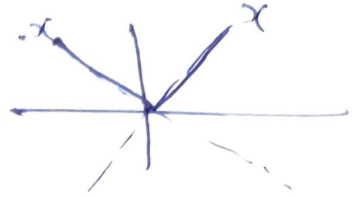
Ans : $v_2 = 0$, $-1 \leq v \leq 1$

(C) Let $f_1(x) = -x$ and $f_2(x) = x$, $x \in R$. Use the result $\partial(\max f_i)(\bar{x}) = Co \bigcup_{i \in I(\bar{x})} \partial f_i(\bar{x})$, $i = 1, 2$. (f_1, f_2 are cx fun) to compute $\partial f(0)$.

Sol:

$$f(x) = \max\{f_1(x), f_2(x)\}$$

$$= |x|, \quad x \in R$$



At $\bar{x} = 0$:

$$f_1(\bar{x}) = 0$$
$$f_2(\bar{x}) = 0$$

and

$$f(0) = \max(0, 0) = 0$$

$$I(\bar{x}) = \{1, 2\}$$

$$\partial f_1(\bar{x}) = \{-1\}$$

$$\partial f_2(\bar{x}) = \{1\}$$

Therefore

$$\partial f(\bar{x}) = \partial(\max(f_1, f_2)) = Co\{\partial f_1(\bar{x}) \cup \partial f_2(\bar{x})\}$$

$$= Co\{-1, 1\}$$

$$= [-1, 1]$$

OR

... extended real valued function ... at a point, then show that

OR

c) Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a function and let $\bar{x} \in \text{int}(\text{dom} f)$. Gateaux derivative of f at $\bar{x}$ is denoted by $f'_G(x)$ and defined as

$$\langle f'_G(\bar{x}, d), d \rangle = \lim_{t \to 0} \frac{f(\bar{x} + td) - f(\bar{x})}{t}$$

Suppose f is Gateaux diff. at $\bar{x}$. Let $V = f'_G(x)$. Then

$$f'(\bar{x}, d) = \langle V, d \rangle = \underline{f}'(\bar{x}; d) \quad \forall d \in \mathbb{R}^n \quad -(1)$$

For any $w \in \partial f(\bar{x})$, Then

$$\underline{f}'(\bar{x}; d) \leq \langle w, d \rangle \leq f'(\bar{x}; d) \quad \forall d \in \mathbb{R}^n \quad -(2)$$

~~(1)~~ →

$$\langle w, d \rangle = \langle V, d \rangle \quad \forall \in \mathbb{R}^n.$$

$$\langle w - V, d \rangle = 0 \quad \forall d \in \mathbb{R}^n \quad (d \neq 0)$$

$$\Rightarrow \quad w - V = 0$$

$$\text{or } w = V = f'_G(\bar{x})$$

Therefore $\partial f(\bar{x}) = \{ f'_G(x) \}$
$\qquad\qquad\qquad\qquad\qquad\;\; \| $
$\qquad\qquad\qquad\qquad\qquad\;\; V$

Q.6 (10pts)

(a) Define Fenchel conjugate of an extended real valued function. Compute the Fenchel conjugate of

$$f(x) = \begin{cases} \frac{x^p}{p} & \text{if } x \geq 0 \\ \infty & \text{if } x < 0 \end{cases} \qquad x \in \mathbb{R},\ p \in \mathbb{R},\ p > 1.$$

Sol:

$$f^*(v) = \sup\{vx - f(x) : x \in \mathbb{R}\}$$

$$= \sup\{vx - f(x) : x \in \operatorname{dom} f = [0, \infty)\}$$

$$= \sup\left\{vx - \frac{x^p}{p} : x \geq 0\right\}$$

Let $\psi(x) = vx - \frac{x^p}{p}$ $\quad (v \in \mathbb{R})$

$\psi'(x) = v - x^{p-1}$, $\quad \psi'(x) = 0$ iff $x^{p-1} = v$

(i) $v \geq 0$:

Solving $\psi'(x) = 0$ gives $x = v^{\frac{1}{p-1}}$

x	0		$v^{\frac{1}{p-1}}$		∞
ψ'		$+$	0	$-$	
ψ		↗		↘	

$1 + \frac{1}{p-1} = \frac{p}{p-1}$

$$\psi\left(v^{\frac{1}{p-1}}\right) = v \cdot v^{\frac{1}{p-1}} - \frac{\left(v^{\frac{1}{p-1}}\right)^p}{p}$$

$$= \left(1 - \frac{1}{p}\right) v^{\frac{p}{p-1}} = \left(1 - \frac{1}{p}\right) v^{\frac{1}{\left(1 - \frac{1}{p}\right)}}$$

Let $q > 1$ be defined by $\frac{1}{p} + \frac{1}{q} = 1$.

Then $\psi\left(v^{\frac{1}{p-1}}\right) = \frac{1}{q} v^q$

$$\boxed{f^*(v) = \frac{1}{q} v^q}$$

(ii) $v < 0$:

$$f^*(v) = \sup\Big\{\underbrace{vx - \frac{x^p}{p}}_{\leq 0} : x \geq 0\Big\}$$

$$= 0$$

$$\boxed{f^*(v) = \begin{cases} \frac{v^q}{q} & \text{if } v \geq 0 \\ 0 & \text{if } v < 0 \end{cases}}$$

Q.7 (3+7): Dual cone: Let $\Omega \subseteq R^n$ be a cone.

A dual cone of Ω is the following set

$$\Omega^o = \{p : p^t x \geq 0 \quad \forall x \in \Omega\}$$

A polar cone of Ω is the following set $\Omega^* = \{p : p^t x \leq 0, \forall x \in \Omega\}$

Tangent cone: Let $\bar{x} \in \Omega$, Tangent cone of Ω at $\bar{x}$ denoted by $T(\bar{x}, \Omega)$ or $T_\Omega(\bar{x})$ is defined as follows

$T_\Omega(\bar{x}) =$ closure of feasible direction at $\bar{x}$ = closure $(F_\Omega(\bar{x}))$

(b) Let $v \in N_\Omega(\bar{x})$. Then for any $d \in F_\Omega(\bar{x})$, there exists $\epsilon > 0$ such that $\bar{x} + \epsilon d \in \Omega$. Hence

$$\langle v, d \rangle = \frac{1}{\epsilon} \langle v, \bar{x} + \epsilon d - \bar{x} \rangle \leq 0.$$

For any $d \in T_\Omega(\bar{x})$, there exists a sequence d_n such that $d_n \in F_\Omega(\bar{x})$ and $d_n \to \hat{d}$.

As $\langle v, d_n \rangle \leq 0$, we have $\langle v, \hat{d} \rangle \leq 0$ which implies

$$v \in (T_\Omega(\bar{x}))^* \quad \text{i.e } N_\Omega(\bar{x}) \subseteq (T_\Omega(\bar{x}))^*.$$

On the other hand,

let $v \in (T_\Omega(\bar{x}))^*$. Then $\langle v, d \rangle \leq 0 \quad \forall d \in T_\Omega(\bar{x})$.

For any $x \in \Omega$, ~~and d > 0~~ because Ω is convex set, there exists $d \in T_\Omega(\bar{x})$ and $\alpha > 0$ such that $x = \bar{x} + \alpha d$.

Hence we have

$$\langle v, x - \bar{x} \rangle = \langle v, \alpha d \rangle = \alpha \langle v, d \rangle \leq 0$$

Thus $v \in N_\Omega(\bar{x})$ i.e $(T_\Omega(\bar{x}))^* \subseteq N_\Omega(\bar{x})$.

OR

(b) We know that

$$N(\bar{x}; \Omega_1 \cap \Omega_2) = N(\bar{x}; \Omega_1) + N(\bar{x}; \Omega_2)$$

and $$T(\bar{x}; \Omega) = (N(\bar{x}, \Omega))^*$$

Now
$$\begin{aligned} T(\bar{x}; \Omega_1 \cap \Omega_2) &= [N(\bar{x}; \Omega_1 \cap \Omega_2)]^* \\ &= [N(\bar{x}; \Omega_1) + N(\bar{x}; \Omega_2)]^* \\ &= N(\bar{x}; \Omega_1)^* \cap N(\bar{x}; \Omega_2)^* \quad \text{(by prop)} \\ &= T(\bar{x}; \Omega_1) \cap T(\bar{x}; \Omega_2). \end{aligned}$$

~~(b) State the Slater constraint qualification.~~

~~(c) State the Karush-Kuhn-Tucker necessary conditions for a general nonlinear programming problem.~~ Determine all solutions of the KKT conditions for the following problem.

$$\text{Max} \quad (x_1+2)^2 + (x_2-1)^2$$

$$\text{s/t} \quad -x_1 + x_2 - 2 \le 0$$

$$x_1^2 - x_2 \le 0$$

Sol.

$$\begin{pmatrix} -2(x_1+2) \\ -2(x_2-1) \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2x_1 \\ -1 \end{pmatrix} = 0$$

or

$$\left.\begin{aligned} -2x_1 - 4 - \lambda_1 + 2\lambda_2 x_1 &= 0 \\ -2x_2 + 2 + \lambda_1 - \lambda_2 &= 0 \\ \lambda_1(-x_1+x_2-2) &= 0 \\ \lambda_2(x_1^2 - x_2) &= 0 \end{aligned}\right\} \quad (1)$$

$$-x_1 - x_2 - 2 \le 0, \quad x_1^2 - x_2 \le 0, \quad \lambda_1, \lambda_2 \ge 0 \quad (2)$$

(i) $\lambda_1 = \lambda_2 = 0$:

$$-2x_1 - 4 = 0$$

$$-2x_2 - 2 = 0$$

$$\Rightarrow (x_1, x_2) = (-2, 1)$$

$(-2,1)$ does not satisfy (2)

or $(-2,1)$ is not a feasible point

(ii) $\lambda_1 = 0$, $\lambda_2 > 0$:

$$-2x_1 - 4 + 2\lambda_2 x_1 = 0 \quad \text{——} (*)$$

$$-2x_2 + 2 - \lambda_2 = 0$$

$$x_1^2 = x_2$$

$$\Rightarrow \lambda_2 = 2 - 2x_2$$

$$= 2(1 - x_1^2)$$

Substituting λ_2 in $(*)$, we get

$$x_1^3 = \frac{x_1}{2} - 1 \quad \text{——} (**)$$

only real solution $x_1 < -1$ satisfies $(**)$

we get $\lambda_2 < 0$

contradicts

(iii) $\lambda_1 > 0$, $\lambda_2 = 0$:

$$\left.\begin{aligned} -2x_1 - 4 - \lambda_1 &= 0 \\ -2x_1 + 2 + \lambda_1 &= 0 \\ -x_1 + x_2 - 2 &= 0, \\ x_1^2 - x_2 &\le 0 \\ \lambda_1 &> 0 \end{aligned}\right] (\$)$$

unique solution $x_1 = -\frac{3}{2}$, $x_2 = \frac{1}{2}$, $\lambda_2 = -1$

does not satisfy (\$)

(iv) $\lambda_1, \lambda_2 > 0$:

$$-2x_1 - 4 - \lambda_1 + 2x_1\lambda_2 = 0$$

$$-2x_2 + 2 + \lambda_1 - \lambda_2 = 0$$

$$-x_1 + x_2 = 2$$

$$x_1^2 = x_2$$

$$\lambda_1, \lambda_2 > 0$$

$-x_1 + x_1^2 = 2$
$x_1(1 - x_1) = 2$
$x_1 = 2$ or $x_1 = -1$

$$\Rightarrow x_1 = 2 \quad \text{and} \quad x_2 = 4$$

$$\text{or } x_1 = -1 \quad \text{and} \quad x_2 = 1$$

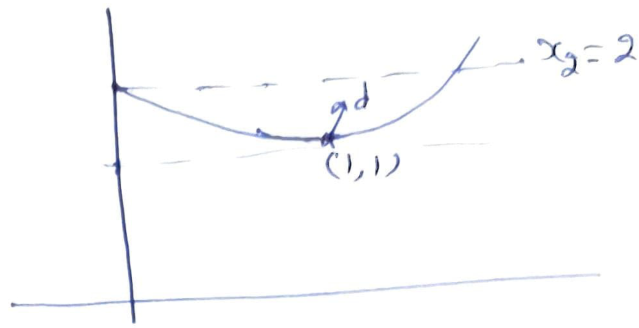
For $x_1 = -1$, $x_2 = 1$, we get $\lambda_1 = \lambda_2 = -\frac{2}{3}$ (Not satisfied)

$x_1 = 2$, $x_2 = 4$, $\lambda_1 = \frac{32}{3}$, $\lambda_2 = \frac{14}{3}$

$$\boxed{(x_1, x_2) = (2, 4)}$$

Q 9 (10 pts)

$S = \{x \in R^2 ; x_2 - 2 \leq 0, 1 + (x_1 - 1)^2 - x_2 \leq 0\}$



$\bar{x} + \lambda d = (1,1) + \lambda (d_1, d_2)$

$= (1 + \lambda d_1, 1 + \lambda d_2)$

$x_2 - 2 \leq 0 \Rightarrow 1 + \lambda d_2 - 2 \leq 0$

$\Rightarrow \lambda d_2 \leq 1.$

$1 + (x_1 - 1)^2 - x_2 \leq 0 \Rightarrow 1 + (1 + \lambda d_1 - 1)^2 - 1 - \lambda d_2 \leq 0$

$\Rightarrow \lambda^2 d_1^2 - \lambda d_2 \leq 0$

$\Rightarrow d_2 \geq \lambda d_1^2$

$F_D(1,1) = \{d \in R^2 : d_2 > 0\}$

$d_2 > 0, d_1 \in R \setminus \{0\}$

[redacted]