KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DEPARTMENT OF MATHEMATICS

Comprehensive Exam, Term-251

Course Code: MATH 581 **Course Title: Advanced Linear Programming**

Name: ____________________________

KFUPM ID: _______________________

Question	Max. Grade	Grade	Comments
PART I: Solve ALL questions in this part			
1	10		
2	10		
3	10		
PART II: Solve 5 questions out of the 7 questions in this part			
4	14		
5	14		
6	14		
7	14		
8	14		
9	14		
10	14		
TOTAL	**100**		

Recommendation:

- Justify your answers thoroughly.

1. Write all steps of *Karmarkar's algorithm* for the standard Karmarkar canonical form of a linear program.

Solution: Canonical form and notation.

$$\min\ c^\top x \quad \text{s.t.} \quad Ax = 0,\ \ e^\top x = 1,\ \ x \geq 0,$$

with $x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ (full row rank), and $e = (1, \dots, 1)^\top$.

Step 1 (Initialization). Choose a strictly feasible point

$$x^{(0)} = \frac{1}{n}e \quad (\text{so } Ax^{(0)} = 0,\ e^\top x^{(0)} = 1,\ x^{(0)} > 0),$$

and set $k = 0$.

Step 2 (Stopping test). If $c^\top x^{(k)} \leq \varepsilon$ ("sufficiently close to zero"), stop. In that case $x^{(k)}$ can be (optimally) rounded to a basic feasible solution $\bar{x}$. Otherwise, continue.

Step 3 (Projection). Let

$$D_R = \operatorname{diag}\big(x^{(k)}\big), \qquad B = \begin{bmatrix} AD_R \\ e^\top \end{bmatrix} \in \mathbb{R}^{(m+1)\times n}.$$

Project $D_R c$ onto the subspace $\{y : AD_R y = 0,\ e^\top y = 0\}$:

$$c_p \ = \ \Big(I - B^\top (BB^\top)^{-1} B\Big)\,(D_R c).$$

Step 4 (Centered step and projective map). Fix $\alpha \in (0, 1)$ (take $\alpha = \frac{1}{4}$ for convergence). Update in the scaled (centered) variables:

$$y^{(k+1)} \ = \ \frac{1}{n}e \ - \ \frac{\alpha}{\sqrt{n(n-1)}}\,\frac{c_p}{\|c_p\|_2}.$$

Map back to the original variables via the projective transformation:

$$x^{(k+1)} \ = \ \Phi\big(y^{(k+1)}\big) \ = \ \frac{D_R\, y^{(k+1)}}{e^\top D_R\, y^{(k+1)}}.$$

Step 5 (Iterate). Set $k \leftarrow k + 1$ and return to Step 2.

2. State and prove the *Weak Duality Theorem* for linear programming. Then write the dual of the following problem:

$$\begin{aligned} &\min\ z = 8x_1 + 5x_2 + 4x_3 \\ \text{s.t.}\quad & 4x_1 + 2x_2 + 8x_3 = 12, \\ & 7x_1 + 5x_2 + 6x_3 \geq 9, \\ & 8x_1 + 5x_2 + 4x_3 \leq 10, \\ & x_1 \geq 0, \quad x_2 \text{ is unrestricted in sign}, \quad x_3 \leq 0. \end{aligned}$$

Solution: Weak Duality Theorem. For any primal–dual pair

$$(P):\ \min\{c^\top x : Ax \geq b,\ x \geq 0\}, \qquad (D):\ \max\{b^\top y : A^\top y \leq c,\ y \geq 0\},$$

if x is feasible for (P) and y is feasible for (D), then

$$c^\top x\ \geq\ b^\top y.$$

Proof. From feasibility, $Ax \geq b$ and $A^\top y \leq c$ with $x \geq 0$, $y \geq 0$. Then

$$b^T y = y^T b \leq y^T(Ax) = (A^T y)^T x \leq c^t x.$$

This proves weak duality. □

Dual of the given problem. Primal formulation:

$$\begin{aligned} &\min\ z = 8x_1 + 5x_2 + 4x_3 \\ \text{s.t.}\quad & 4x_1 + 2x_2 + 8x_3 = 12, \\ & 7x_1 + 5x_2 + 6x_3 \geq 9, \\ & 8x_1 + 5x_2 + 4x_3 \leq 10, \\ & x_1 \geq 0, \quad x_2 \text{ free}, \quad x_3 \leq 0. \end{aligned}$$

Dual problem:

$$\begin{aligned} &\max\ \omega = 12y_1 + 9y_2 + 10y_3 \\ \text{s.t.}\quad & 4y_1 + 7y_2 + 8y_3 \leq 8, \\ & 2y_1 + 5y_2 + 5y_3 = 5, \\ & 8y_1 + 6y_2 + 4y_3 \geq 4, \\ & y_1 \text{ free}, \quad y_2 \leq 0, \quad y_3 \geq 0. \end{aligned}$$

3. A machine tool company decides to make four subassemblies through four contractors. Each contractor is to receive only one subassembly. The cost (hundreds of riyals) is given below. Assign the subassemblies to contractors so as to minimize the total cost.

	I	II	III	IV
A	15	13	14	17
B	11	12	15	13
C	13	12	10	11
D	15	17	14	16

Solution: Given cost matrix (hundreds of riyals):

$C =$

	I	II	III	IV
A	15	13	14	17
B	11	12	15	13
C	13	12	10	11
D	15	17	14	16

Step 1 (Row reduction). Subtract the minimum of each row: $r_A = 13,\ r_B = 11,\ r_C = 10,\ r_D = 14$.

$C^{(1)} =$

	I	II	III	IV
A	2	0	1	4
B	0	1	4	2
C	3	2	0	1
D	1	3	0	2

Step 2 (Column reduction). Column IV has no zero; subtract its minimum 1 from column IV.

$C^{(2)} =$

	I	II	III	IV
A	2	0	1	3
B	0	1	4	1
C	3	2	0	0
D	1	3	0	1

Now there are independent zeros that allow a full assignment.

Step 3 (Choose independent zeros). One valid choice:

$$(A, \text{II}),\ (B, \text{I}),\ (C, \text{IV}),\ (D, \text{III}).$$

This gives the unique contractor for each subassembly with no conflicts.

Step 4 (Compute total cost using the original matrix).

$$\text{Cost} = C_{A,\text{II}} + C_{B,\text{I}} + C_{C,\text{IV}} + C_{D,\text{III}} = 13 + 11 + 11 + 14 = 49 \text{ (hundreds)}.$$

Thus the minimum total cost is $49 \times 100 = \boxed{4900 \text{ riyals}}$.

Optimal assignment: $A \to \text{II},\ B \to \text{I},\ C \to \text{IV},\ D \to \text{III}.$

4. Consider the linear fractional program

$$\min \frac{2x_1 + 3x_2}{x_1 + x_2 + 1} \quad \text{s.t. } 2x_1 + x_2 \le 3, \quad x_1 + 2x_2 \le 3, \quad x_1, x_2 \ge 0.$$

Convert this problem to a linear program using the Charnes–Cooper transformation and then solve it.

Solution: We are asked to solve the fractional program

$$\min \frac{2x_1 + 3x_2}{x_1 + x_2 + 1} \quad \text{s.t. } 2x_1 + x_2 \le 3, \; x_1 + 2x_2 \le 3, \; x_1, x_2 \ge 0.$$

Step 1. Charnes–Cooper transformation. Let $y_i = x_i/(x_1 + x_2 + 1)$ for $i = 1, 2$ and $t = 1/(x_1 + x_2 + 1)$. Then $y_1 + y_2 + t = 1$, $y_1, y_2, t \ge 0$. The objective becomes

$$\min Z = 2y_1 + 3y_2.$$

Constraints transform to

$$2y_1 + y_2 \le 3t,$$
$$y_1 + 2y_2 \le 3t.$$

Step 2. Standard form LP.

$$\begin{aligned} \min\ & 2y_1 + 3y_2 \\ \text{s.t. } & 2y_1 + y_2 - 3t \le 0, \\ & y_1 + 2y_2 - 3t \le 0, \\ & y_1 + y_2 + t = 1, \\ & y_1, y_2, t \ge 0. \end{aligned}$$

Step 3. Two–phase simplex (from the tableau shown). - Phase I yields feasibility. - Phase II optimization gives the final tableau with basic feasible solution

$$y_1 = 0, \; y_2 = \tfrac{3}{5}, \; t = \tfrac{2}{5}.$$

Step 4. Recover original variables. Since $t = 2/5$, we get

$$x_1 = y_1/t = 0, \quad x_2 = (3/5)/(2/5) = \tfrac{3}{2}.$$

Objective value in the original problem:

$$Z^* = \frac{2(0) + 3(3/2)}{0 + (3/2) + 1} = \frac{9/2}{5/2} = \tfrac{9}{5} = 1.8.$$

Final Answer: The optimal solution is $x_1 = 0, \; x_2 = \frac{3}{2}$ with minimum value $Z^* = 1.8$.

5. Solve by the *Revised Simplex Method*:

$$\begin{aligned} \max\ z &= x_1 + 2x_2 \\ \text{s.t.}\quad x_1 + x_2 &\le 3, \\ x_1 + 2x_2 &\le 5, \\ 3x_1 + x_2 &\le 6, \\ x_1, x_2 &\ge 0. \end{aligned}$$

Solution: **Problem.** Maximize $z = x_1 + 2x_2$ subject to

$$\begin{aligned} x_1 + x_2 &\le 3, \\ x_1 + 2x_2 &\le 5, \\ 3x_1 + x_2 &\le 6, \\ x_1, x_2 &\ge 0. \end{aligned}$$

Introduce slacks $s_1, s_2, s_3 \ge 0$ to obtain $x_1 + x_2 + s_1 = 3$, $x_1 + 2x_2 + s_2 = 5$, $3x_1 + x_2 + s_3 = 6$.

Initial data. Basic variables $B = \{s_1, s_2, s_3\}$ with $B^{-1} = I$ and $x_B = b = (3, 5, 6)^\top$. Cost vectors: $c_B = (0, 0, 0)$, $c_N = (1, 2, 0, 0)$ for (x_1, x_2). Reduced costs (using $\bar{c}_j = c_j - c_B^\top B^{-1} a_j$) are

$$\bar{c}_{x_1} = 1 - 0 = 1, \qquad \bar{c}_{x_2} = 2 - 0 = 2,$$

so with the sign convention in the working notes ($\Delta_j = -\bar{c}_j$), we have $\Delta_1 = -1$, $\Delta_2 = -2$ and the most negative Δ_2 selects x_2 as the entering variable.

Iteration 1. Compute the direction column

$$X_R = B^{-1} a_{x_2} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mapsto \text{(after the row operations shown)} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

With current $x_B = (3, 5, 6)^\top$, the ratio test on positive components of X_R gives

$$\frac{3}{1}, \frac{5}{2}, \frac{6}{1} \quad \Rightarrow \quad \min = \frac{5}{2},$$

so s_2 leaves and x_2 enters.

Perform the pivot (as in the tableau), yielding the new basis $B = \{s_1, x_2, s_3\}$. The updated inverse (from the notes) gives

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}, \qquad x_B = B^{-1} b = \begin{bmatrix} \frac{1}{2} \\ \frac{5}{2} \\ \frac{7}{2} \end{bmatrix}.$$

Optimality test. Using the first row of the revised tableau (or $c_B^\top B^{-1}A - c^\top$), the reduced costs satisfy $\bar{c}_{x_1} \geq 0$ and $\bar{c}_{s_i} \geq 0$ (equivalently $\Delta_j \geq 0$ in the notes). Hence the basic solution is optimal.

Solution.

$$x_1 = 0, \qquad x_2 = \tfrac{5}{2}, \qquad s_1 = \tfrac{1}{2},\ s_2 = 0,\ s_3 = \tfrac{7}{2}.$$

Objective value

$$z = x_1 + 2x_2 = 0 + 2 \cdot \tfrac{5}{2} = \boxed{5}.$$

6. Solve the following problem by the *Branch-and-Bound* method:

$$\max\ z = 7x_1 + 9x_2$$
$$\begin{aligned}\text{s.t.}\quad -x_1 + 3x_2 &\le 6,\\ 7x_1 + x_2 &\le 35,\\ x_1 &\le 7,\\ x_2 &\le 7,\\ x_1, x_2 &\ge 0, \quad x_1, x_2 \text{ are integers.}\end{aligned}$$

(LP relaxation solution: $x_1 = 4.5,\ x_2 = 3.5,\ \max z = 63.$*)*

Solution: Problem. Maximize $z = 7x_1 + 9x_2$ subject to

$$-x_1 + 3x_2 \le 6, \quad 7x_1 + x_2 \le 35, \quad x_1 \le 7,\ x_2 \le 7, \quad x_1, x_2 \in \mathbb{Z}_{\ge 0}.$$

(Noninteger relaxation solution provided: $x_1 = 4.5,\ x_2 = 3.5,\ z = 63.$)

1) Branching choice. The LP optimum has both variables fractional. Choose to branch on x_1:

$$\text{S-1: } x_1 \le 4, \qquad \text{S-2: } x_1 \ge 5.$$

2) Node S-1 ($x_1 \le 4$). Solve the LP relaxation. With $x_1 \le 4$, the active pair $-x_1 + 3x_2 \le 6$ and $x_1 \le 4$ gives

$$(x_1, x_2) = (4, \tfrac{10}{3}), \quad z = 7 \cdot 4 + 9 \cdot \tfrac{10}{3} = 28 + 30 = \boxed{58} \quad \text{(upper bound)}.$$

Since x_2 is fractional, branch on x_2:

$$\text{S-1a: } x_2 \le 3, \qquad \text{S-1b: } x_2 \ge 4.$$

S-1a ($x_2 \le 3$). The LP extreme point occurs at $(x_1, x_2) = (4, 3)$, which is integer and feasible;

$$z = 7 \cdot 4 + 9 \cdot 3 = \boxed{55}.$$

Record incumbent $z^* = 55$.

S-1b ($x_2 \ge 4$). From $-x_1 + 3x_2 \le 6$ with $x_1 \ge 0$ we have $3x_2 \le 6 \Rightarrow x_2 \le 2$, contradicting $x_2 \ge 4$. Hence S-1b is **infeasible** and is fathomed.

3) Node S-2 ($x_1 \ge 5$). From $7x_1 + x_2 \le 35$ we get $x_2 \le 35 - 7x_1 \le 0$ (since $x_1 \ge 5$). Thus the only feasible point in the relaxation is $(x_1, x_2) = (5, 0)$ with

$$z = 7 \cdot 5 + 9 \cdot 0 = \boxed{35} < z^*.$$

Node S-2 is fathomed by bound.

Conclusion. All branches are fathomed; the best incumbent is

$$\boxed{x_1 = 4, \quad x_2 = 3, \quad z = 55},$$

which is the optimal integer solution.

7. Solve the following integer program by the *Cutting-plane* method:

$$\begin{aligned} \max\ z &= x_1 - 3x_2 \\ \text{s.t.}\quad x_1 - x_2 &\le 2, \\ 2x_1 + 4x_2 &\le 15, \\ x_1, x_2 &\ge 0,\ \text{integers.} \end{aligned}$$

The optimal (LP) tableau of the relaxation is:

	x_1	x_2	s_1	s_2	RHS
s_1	$\frac{3}{2}$	0	1	$\frac{1}{4}$	$\frac{23}{4}$
x_2	$\frac{1}{2}$	1	0	$\frac{1}{4}$	$\frac{15}{4}$
$z_j - c_j$	$\frac{5}{2}$	0	0	$\frac{3}{4}$	$\frac{45}{4}$

Solution: Maximization version. Maximize $z = x_1 - 3x_2$ subject to

$$x_1 - x_2 \le 2, \qquad 2x_1 + 4x_2 \le 15, \qquad x_1, x_2 \in \mathbb{Z}_{\ge 0}.$$

LP relaxation. Ignore integrality and solve

$$\max\ z = x_1 - 3x_2 \quad \text{s.t.} \quad x_1 - x_2 \le 2,\ x_1 + 2x_2 \le 7.5,\ x_1, x_2 \ge 0.$$

A quick bound shows optimality at an extreme point: from $x_1 - x_2 \le 2$ we get

$$z = x_1 - 3x_2 \ \le\ (x_2 + 2) - 3x_2 \ =\ 2 - 2x_2 \ \le\ 2 \quad (\text{since } x_2 \ge 0).$$

The bound $z \le 2$ is attained at $(x_1, x_2) = (2, 0)$ (which also satisfies $x_1 + 2x_2 \le 7.5$). Hence the LP relaxation optimum is

$$(x_1, x_2) = (2, 0), \qquad z^* = 2.$$

Integrality / Cutting-plane check. The LP optimum $(2, 0)$ is already integral, so no Gomory cut is needed; the cutting-plane procedure would stop immediately at this node.

Conclusion. For the *maximization* formulation of Q7 the optimal integer solution is

$$\boxed{x_1 = 2,\ x_2 = 0,\ z = 2},$$

coinciding with the LP-relaxation optimum.

8. (a) Show that the feasible region of a linear programming problem is a convex set.

(b) Use the complementary slackness conditions to solve the following primal problem

$$\min\ z = 2x_1 + 3x_2 + 5x_3 + 2x_4 + 3x_5$$
$$\text{s.t.}\quad x_1 + x_2 + 2x_3 + x_4 + 3x_5 \geq 4,$$
$$2x_1 - 2x_2 + 3x_3 + x_4 + x_5 \geq 3,$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0,$$

given that an optimal dual solution is $\left(\frac{4}{5}, \frac{3}{5}\right)$.

Solution: (a) Convexity of the feasible region. The feasible set of an LP is an intersection of halfspaces/hyperplanes, each convex; intersections of convex sets are convex. Hence the region is convex.

(b) Dual program is

$$\begin{aligned} \max \quad & 4y_1 + 3y_2 \\ \text{subject to} \quad & y_1 + 2y_2 \leq 2, \\ & y_1 - 2y_2 \leq 3, \\ & 2y_1 + 3y_2 \leq 5, \\ & y_1 + y_2 \leq 2, \\ & 3y_1 + y_2 \leq 3, \\ & y_1, y_2 \geq 0. \end{aligned}$$

$$\text{Solution is } y_1 = \tfrac{4}{5},\ y_2 = \tfrac{3}{5}.$$

Applying CSS, we find in the dual constraints:

$$\frac{4}{5} + \frac{6}{5} = 2 \ \checkmark$$
$$\frac{4}{5} - \frac{6}{5} = -\tfrac{2}{5} < 3 \quad \Rightarrow x_2 = 0$$
$$\frac{8}{5} + \frac{9}{5} = \tfrac{17}{5} < 5 \quad \Rightarrow x_3 = 0$$
$$\frac{4}{5} + \frac{3}{5} = \tfrac{7}{5} < 2 \quad \Rightarrow x_4 = 0$$
$$\frac{12}{5} + \frac{3}{5} = 3$$

Since $y_1, y_2 > 0$, then

$$x_1 + x_2 + 2x_3 + x_4 + 3x_5 = 4,$$
$$2x_1 - 2x_2 + 3x_3 + x_4 + x_5 = 3.$$

Given $x_2 = x_3 = x_4 = 0$, i.e.

$$x_1 + 3x_5 = 4,$$
$$2x_1 + x_5 = 3.$$

$$\Rightarrow x_1 = 1,\ x_5 = 1.$$

Optimal solution is $(1, 0, 0, 0, 1)$.

9. Consider the linear program

$$\begin{aligned} \max\ z &= 6x_1 + 8x_2 \\ \text{s.t.}\quad 5x_1 + 10x_2 &\le 60, \\ 4x_1 + 4x_2 &\le 40, \\ x_1, x_2 &\ge 0. \end{aligned}$$

An optimal tableau is:

	x_1	x_2	s_1	s_2	RHS
x_2	0	1	$\frac{1}{5}$	$-\frac{1}{4}$	2
x_1	1	0	$-\frac{1}{5}$	$\frac{1}{2}$	8
$z_j - c_j$	0	0	$\frac{2}{5}$	1	64

Apply sensitivity analysis to determine the optimal solution if the RHS vector $\begin{bmatrix} 60 \\ 40 \end{bmatrix}$ is changed to $\begin{bmatrix} 20 \\ 40 \end{bmatrix}$.

Solution: We are given the optimal tableau for

$$\max z = 6x_1 + 8x_2 \quad \text{s.t.} \quad \begin{cases} 5x_1 + 10x_2 \le 60, \\ 4x_1 + 4x_2 \le 40, \\ x_1, x_2 \ge 0, \end{cases}$$

whose final (optimal) basis is $B = \{x_2, x_1\}$ with

$$B^{-1} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{4} \\ -\frac{1}{5} & \frac{1}{2} \end{bmatrix}, \qquad x_B = B^{-1}b = \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \qquad z = 64.$$

Now the right–hand side is changed to $b' = (20, 40)^\top$. Use sensitivity/dual simplex to reoptimize.

1) Check feasibility of the current basis under b'.

$$x'_B \ = \ B^{-1}b' \ = \begin{bmatrix} \frac{1}{5} & -\frac{1}{4} \\ -\frac{1}{5} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 20 \\ 40 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 16 \end{bmatrix}.$$

Since $x'_2 = -6 < 0$, the basic solution is *infeasible*; apply the *dual simplex method* (reduced costs were already optimal for the original problem).

2) Dual simplex pivot. Choose the most negative basic value as the leaving row: the x_2-row. In that row, consider columns with negative coefficients; compute the ratios $\frac{z_j - c_j}{a_{rj}}$ and select the minimum to enter. Performing the indicated pivot (as in the working) yields a new basis with s_2 entering and x_2 leaving.

After one pivot, the tableau becomes feasible with basic variables $B = \{s_2, x_1\}$ and right–hand side

$$x_B = \begin{bmatrix} 24 \\ \\ 4 \end{bmatrix} \quad \text{(i.e., } s_2 = 24,\ x_1 = 4\text{)},$$

while $x_2 = 0$. The reduced-cost row satisfies $z_j - c_j \geq 0$, so optimality holds.

3) Answer. With $b' = (20, 40)$ the optimal solution is

$$x_1 = 4, \qquad x_2 = 0, \qquad z = 6 \cdot 4 + 8 \cdot 0 = \boxed{24}.$$

10. (a) Find an initial basic feasible solution for the following transportation problem using Vogel's Approximation Method (VAM):

	D1	D2	D3	D4	Supply
S1	10	2	20	11	15
S2	12	7	9	20	25
S3	4	14	16	18	10
Demand	5	15	15	15	

(b) Write the dual of the standard transportation problem and outline all the steps of the u–v method to test optimality and improve a basic feasible solution.

Solution:

(a) Vogel's Approximation Method (VAM)

Data. The transportation tableau is

$$C = \begin{bmatrix} 10 & 2 & 20 & 11 \\ 12 & 7 & 9 & 20 \\ 4 & 14 & 16 & 18 \end{bmatrix}, \qquad \text{Supply } (15, 25, 10), \quad \text{Demand } (5, 15, 15, 15).$$

It is already balanced.

Penalties (first iteration).

- Row penalties: $S_1 : 10 - 2 = 8$, $S_2 : 9 - 7 = 2$, $S_3 : 14 - 4 = 10$.
- Column penalties: $D_1 : 10 - 4 = 6$, $D_2 : 7 - 2 = 5$, $D_3 : 16 - 9 = 7$, $D_4 : 18 - 11 = 7$.
- Largest penalty $= 10$ (row S_3). In row S_3, the least cost is $c_{31} = 4$; allocate $\min(10, 5) = 5$ to $(3, 1)$. Update: $S_3 \to 5$, $D_1 \to 0$.

Second iteration. With D_1 satisfied, recompute penalties; largest is 8 (row S_1). In S_1, least cost is $c_{12} = 2$; allocate $\min(15, 15) = 15$ to $(1, 2)$. Update: $S_1 \to 0$, $D_2 \to 0$.

Third iteration. Remaining demands: $D_3 = 15$, $D_4 = 15$; supplies: $S_2 = 25$, $S_3 = 5$. Choose column D_4 (largest penalty tie). Least cost among remaining rows is at S_3 (cost 18); allocate $\min(5,15) = 5$ to $(3,4)$. Update: $S_3 \to 0$, $D_4 \to 10$.

Finish. Only S_2 has supply left (25), and the remaining demands are $D_3 = 15$, $D_4 = 10$; allocate $x_{23} = 15$ and $x_{24} = 10$.

Initial BFS produced by VAM.

	D_1	D_2	D_3	D_4	Supply
S_1	0	$\boxed{15}$	0	0	15
S_2	0	0	$\boxed{15}$	$\boxed{10}$	25
S_3	$\boxed{5}$	0	0	$\boxed{5}$	10
Demand	5	15	15	15	

Total transportation cost

$$\begin{aligned} \text{TC} &= 15 \cdot 2 + 15 \cdot 9 + 10 \cdot 20 + 5 \cdot 4 + 5 \cdot 18 \\ &= 30 + 135 + 200 + 20 + 90 \\ &= \boxed{475}. \end{aligned}$$

(b) Dual of the standard transportation model and the u-v (MODI) method

Primal (balanced TP, minimization).

$$\begin{aligned} \min \ & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \text{s.t.} \ & \sum_{j=1}^{n} x_{ij} = a_i \quad (i = 1, \ldots, m), \\ & \sum_{i=1}^{m} x_{ij} = b_j \quad (j = 1, \ldots, n), \\ & x_{ij} \geq 0. \end{aligned}$$

Dual (potentials). Introduce u_i for rows and v_j for columns (both unrestricted in sign):

$$\begin{aligned} \max \ & \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j \\ \text{s.t.} \ & u_i + v_j \leq c_{ij} \quad (i = 1, \ldots, m; \ j = 1, \ldots, n). \end{aligned}$$

u-v (MODI) optimality test and improvement.

1. Start with any BFS (e.g., the VAM solution above).
2. Compute potentials: fix one potential (e.g., $u_1 = 0$) and solve $u_i + v_j = c_{ij}$ on all basic cells to get all u_i, v_j.
3. For each nonbasic cell, compute the reduced cost $r_{ij} = c_{ij} - (u_i + v_j)$ (equivalently $d_{ij} = u_i + v_j - c_{ij} = -r_{ij}$).
   - If all $r_{ij} \geq 0$ (all $d_{ij} \leq 0$), the BFS is optimal.
   - Otherwise choose the cell with most negative r_{ij} (most positive d_{ij}) to enter the basis.
4. Form the closed loop through basic cells with alternating $+/-$ signs, take θ as the minimum allocation on the $-$ positions (leaving variable), and update shipments.
5. Repeat steps 2–4 until optimality.