King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 601
Comprehensive Exam– 2020–2021 (203)
August 16, 2021

Allowed Time: 150 minutes

Name: ____

ID #: __________________________

Section #: ______________ Serial Number: ______________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justification !

Question #	Grade	Maximum Points
1		**13**
2		**12**
3		**20**
4		**20**
5		**25**
6		**10**
Total:		**100**

Exercise 1:

A-(8) Let $N(t)$, $t \geq 0$ be a Poisson process with parameter λ. Show that $\lim_{t\longrightarrow\infty} \frac{N(t)}{t} = \lambda$ a.s.

Hint: You may use the Strong law of large numbers : $\lim_{n\longrightarrow\infty} \frac{N(n)}{n} = \lambda$ a.s.

We can write $N(n) = N(1) + (N(2) - N(1)) + \cdots + (N(n) - N(n-1))$

where $N(1), N(2) - N(1), \ldots,$ is a sequence of independent identically distributed r.v.s with expectation: $E(N(1)) = E(N(2) - N(1)) = \cdots = \lambda$

By the Strong law of large numbers:

$$\lim_{n\to\infty} \frac{N(n)}{n} = \lambda \text{ a.s}$$

Now if $n \leq t \leq n+1$, then $N(n) \leq N(t) \leq N(n+1)$ and

$$\frac{N(n)}{n+1} \leq \frac{N(t)}{t} \leq \frac{N(n+1)}{n},$$

But $\lim_{n\to\infty} \frac{N(n)}{n+1} = \lim_{n\to\infty} \frac{N(n+1)}{n} = \lambda$ a.s

Hence $\lim_{t\to\infty} \frac{N(t)}{t} = \lambda$ a.s

B-(5) Let X and Y be two random variables with a Poisson distributions with parameters λ and μ respectively. If X and Y are independent, find the distribution of the random variable $X + Y$.

Using moments generating functions we have:

$M_X(t) = e^{\lambda(e^t - 1)}$ and $M_Y(t) = e^{\mu(e^t-1)}$, Therefore

X and Y are independent

$$M_{X+Y}(t) = M_X(t) M_Y(t) = e^{\lambda(e^t-1)} e^{\mu(e^t-1)} = e^{(\lambda+\mu)(e^t-1)}.$$

Hence $X+Y$ has a Poisson distribution with parameter $\lambda + \mu$. 17

Exercise 2:

Consider the geometric Brownian motion given by $X_t = e^{\mu t + \sigma B_t}$, $t \geq 0$, $\sigma > 0$, $\mu \in \mathbb{R}$.

1-(8) Find $\mathbb{E}(X_t)$ and $Var(X_t)$.

- $E(X_t) = e^{\mu t} E(e^{\sigma B_t}) = e^{\mu t} \cdot e^{\frac{\sigma^2}{2} t} = e^{(\mu + \frac{\sigma^2}{2})t}$
- we Know that for $s \leq t$: $B_t - B_s \overset{d}{=} B_{t-s}$. Hence

$$Cov(X_t, X_s) = E(X_t X_s) - E(X_t)E(X_s) = e^{\mu(t+s)} E\left(e^{\sigma(B_t + B_s)}\right) - e^{(\mu + \frac{\sigma^2}{2})(t+s)}.$$

$$= e^{\mu(t+s)} E\left(e^{\sigma(B_t - B_s) + 2\sigma B_s}\right) - e^{(\mu + \frac{\sigma^2}{2})(t+s)}$$

$$= e^{\mu(t+s)} E\left(e^{\sigma(B_t - B_s)}\right) E\left(e^{2\sigma B_s}\right) - e^{(\mu + \frac{\sigma^2}{2})(t+s)}$$

$$= e^{(\mu + \frac{1}{2}\sigma^2)(t+s)} \left(e^{\sigma^2 s} - 1\right)$$

Hence $Var(X_t) = Cov(X_t, X_t)$

$$= e^{(2\mu + \sigma^2)t} \left(e^{\sigma^2 t} - 1\right).$$

∎

2-(4) Give an application where we can use the geometric Brownian motion X_t.

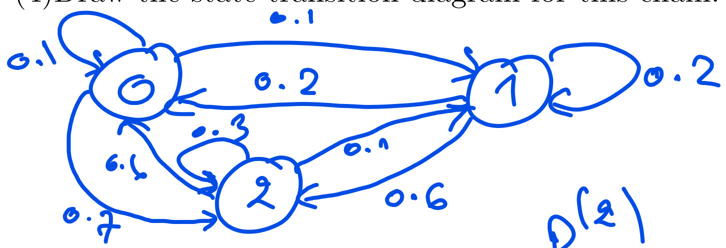
The G.B.M can be used for modelling the stock. prices in the Black-Sholes model.

Exercise 3:

Consider the Markov chain $\{X_n, n \geq 0\}$ with three states, $S = 0, 1, 2$, that has the following transition matrix

$$P = \begin{pmatrix} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.2 & 0.6 \\ 0.6 & 0.1 & 0.3 \end{pmatrix}$$

1. (4)Draw the state transition diagram for this chain.



2. (6) Determine $P(X_3 = 1 \mid X_1 = 0)$. $= p_{01}^{(2)}$.

$$p_{01}^{(2)} = \begin{pmatrix} 0.1 & 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.2 \\ 0.1 \end{pmatrix} = \underline{0.13}$$

Notice that: $P^2 = \begin{pmatrix} 0.47 & \underline{0.13} & 0.4 \\ 0.42 & 0.14 & 0.44 \\ 0.26 & 0.17 & 0.57 \end{pmatrix}$

3. (6) Determine $P(X_3 = 1 \mid X_0 = 0)$. $= p_{01}^{(3)}$, But

$$p_{01}^{(3)} = \begin{pmatrix} 0.1 & 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.2 & 0.6 \\ 0.6 & 0.1 & 0.3 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.2 \\ 0.1 \end{pmatrix} = 1.24$$

4. (4) Assume the initial distributions are $P_0 = P_1 = 0.5$, compute $P(X_2 = 0)$.

$$P(X_2 = 0) = \sum_{j=0}^{2} P_j \, p_{j0}^{(2)} = 0.5\, P_{00}^{(2)} + 0.5\, P_{10}^{(2)}$$

$$= 0.5 \times 0.47 + 0.5 \times 0.42.$$

Exercise 4: Let X_t, Y_t be Itô processes in $\mathbb{R}$.
1)-(8) Prove that:

$$X_t\,Y_t = X_0\,Y_0 + \int_0^t Y_s\,dX_s + \int_0^t X_s\,dY_s + \int_0^t dX_s\,dY_s$$

Apply Itô formula to $g(x,y) = x\cdot y$, gives:

$d(X_t Y_t) = d(g(X_t,Y_t)) = Y_t\,dX_t + X_t\,dY_t + dX_t\,dY_t$.

Hence $X_t Y_t = X_0 Y_0 + \int_0^t Y_s\,dX_s + \int_0^t X_s\,dY_s + \int_0^t dX_s\,dY_s$.

2)-(12) Let $\Phi_t = \exp(-\alpha\,B_t + \frac{1}{2}\,\alpha^2\,t),\ \alpha \in \mathbb{R}$.
i)- Find $d\Phi_t$.
ii)- Given that: $dY_t = r\,dt + \alpha\,Y_t\,dB_t,\ r \in \mathbb{R}$. Prove that $Y_t = Y_0\,\Phi_t^{-1} + r\,\Phi_t^{-1}\int_0^t \Phi_s\,ds$
(**Hint**: Use 1)).

i) By Itô Formula we have:
$d\Phi_t = \Phi_t(-\alpha\,dB_t + \frac{1}{2}\alpha^2 dt) + \frac{1}{2}\Phi_t\alpha^2 dt = (-\alpha\,dB_t + \alpha^2 dt)\cdot\Phi_t$

ii) By 1) we have:
$d(\Phi_t Y_t) = \Phi_t\,dY_t + Y_t\,d\Phi_t + d\Phi_t\,dY_t = \Phi_t\,dY_t + Y_t\Phi_t(-\alpha\,dB_t + \alpha^2 dt)$
$+ (-\alpha\,\Phi_t\,dB_t)(\alpha\,Y_t\,dB_t)$
$= \Phi_t(dY_t - \alpha Y_t\,dB_t) = \Phi_t\,r\,dt$. Integrating we get
$\Phi_t Y_t = \Phi_0 Y_0 + \int_0^t r\,\Phi_s\,ds$. Hence $Y_t = Y_0\Phi_t^{-1} + r\,\Phi_t^{-1}\int_0^t \Phi_s\,ds$

Exercise 5: To describe the motion of a pendulum with small, random perturbations in its environment we consider the stochastic differential equation :

$$U_t'' + \Big(1 + \epsilon W_t\Big) U_t = 0; \qquad U_0,\, U_0' \ \textit{given}, \tag{a}$$

where W_t is a one-dimensional white noise, ϵ a positive constant.

1- (13) Show that the stochastic differential equation (c) can be written in the following form:

$$dX_t = K\, X_t\, dt - \epsilon\, L\, X_t\, dB_t, \tag{b}$$

where X_t, K, L are suitable matrices and B_t a Brownian motion.

Let $X_1(t) = U(t) = U_t$ and $X_2(t) = U_t'$ and $X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$, then

(a) can be written : $X_t' = \begin{pmatrix} X_1'(t) \\ X_2'(t) \end{pmatrix} = \begin{pmatrix} U_t' \\ U_t'' \end{pmatrix} = \begin{pmatrix} X_2(t) \\ -(1+\varepsilon W_t) X_1(t) \end{pmatrix}$

which is interpreted as: $dX_t = \begin{pmatrix} X_2(t) \\ -X_1(t)\,dt - \varepsilon X_1(t)\, dB(t) \end{pmatrix}$

Hence $dX_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} dt - \varepsilon \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} dB_t$

$= K \;\; X_t\, dt - \varepsilon\; L\;\; X_t\, dB_t$.

2- (12) Show that Y_t solves a stochastic Volterra equation of the form

$$Y_t = Y_0 + Y_0'\, t + \int_0^t a(t,r)\, Y_r\, dr + \epsilon \int_0^t \gamma(t,r)\, Y_r\, dB_r\,, \tag{c}$$

where $a(t,r)$ and $\gamma(t,r)$ are functions to be determined.

From 1) we have $y_t' = y'(0) + \int_0^t y''(s)\, ds = y'(0) - \int_0^t y(s)\, ds - \varepsilon \int_0^t y(s)\, dB_s$

Hence, if we Apply the stochastic Fubini theorem :

$y(t) = y(0) + \int_0^t y'(s)\, ds = y(0) + y'(0)t - \int_0^t \left(\int_0^s y(r)\, dr \right) ds - \varepsilon \int_0^t \left(\int_0^s y(r)\, dB_r \right) ds$

$= y(0) + y'(0)t - \int_0^t \left(\int_r^t y(r)\, ds \right) dr - \varepsilon \int_0^t \left(\int_r^t y(r)\, ds \right) dB_r$

$= y(0) + y'(0)t + \int_0^t \underbrace{(r-t)}_{a(t,r)} y(r)\, dr + \varepsilon \int_0^t \underbrace{(r-t)}_{\gamma(t,r)} y(r)\, dB_r$.

Exercise 6: The Black-Scholes-Merton model for growth with uncertain rate of return, is the value of §1 after time t, invested in a saving account. It is described by the following stochastic differential equation:

$$dX_t = \mu X_t\, dt + \sigma X_t\, dB_t,\ \ \mu, \sigma > 0 \tag{d}$$

1-(4) Give the type of the SDE (d).

Linear SDE with multiplicative noise.

2-(6) Prove that the solution of the SDE (d) is given by a Geometric Brownian motion.

Applying Itô's Formula to $g(t,x) = e^{\mu t + \sigma x}$, it gives:

$dX_t = \mu X_t\, dt + \sigma X_t\, dB_t$, hence $X_t = e^{\mu t + \sigma B_t}$ (Geo. B.M) is the solution of the SDE (d).