King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 601

Comprehensive Exam– Term 232

Wednesday, January 24 , 2024

Allowed Time: 150 minutes

Name: ____________________

ID #: ____________________

Section #: ____________ Serial Number: ____________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work.** No points for answers without justifications !

Question #	Grade	Maximum Points
1		15
2		12
3		12
4		15
5		13
6		16
7		17
Total:		100

Exercise 1:(15)

Let X be a continuous random variable that takes only nonnegative values with mean μ and variance σ^2.

A- i)- Prove that for any value $a > 0$

$$P\{X \geq a\} \leq \frac{\mathbb{E}(X)}{a} \qquad (a)$$

$E(X) = \int_0^\infty x f(x)\,dx = \int_0^a x f(x)\,dx + \int_a^\infty x f(x)\,dx \geq \int_a^\infty x f(x)\,dx$

Thus $E(X) \geq \int_a^\infty a f(x)\,dx = a \int_a^\infty f(x)\,dx = a P(X \geq a)$

Hence $P(X \geq a) \leq \frac{E(X)}{a}$.

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ii)- Prove that for any $k > 0$

$$P(\mid X - \mu \mid \geq k) \leq \frac{\sigma^2}{k^2} \qquad (b)$$

Hint: You may use the result in (a)

$Y = (X-\mu)^2$ is a nonnegative random variable, we can Apply Markov inequality (result in a) for $a = k^2$ we get:

04 $P((X-\mu)^2 \geq k^2) \leq \frac{E((X-\mu)^2)}{k^2}$ or $(X-\mu)^2 \geq k^2 \Leftrightarrow |X-\mu| \geq k$,

Hence $P(|X-\mu| \geq k) \leq \frac{E((X-\mu)^2)}{k^2} = \frac{\sigma^2}{k^2}$

B- Suppose we know that the number of items produced in a factory during a week is a random variable with mean 500.

1- Find the probability that this weeks production will be at least 1000 ?

Let X be the number of items that will be produced in a week.

a) By Markov's inequality a) $P(X \geq 1000) \leq \frac{E(X)}{1000} = \frac{500}{1000} = \frac{1}{2}$,

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2- If the variance of a week's production is equal 100, then find the probability that this week's production will be between 400 and 600 ?

Hint: You may use the results in **A**).

By Chebyshev's inequality (b): $P(|X-500| \geq 100) \leq \frac{\sigma^2}{(100)^2} = \frac{1}{100}$ 03

Hence $P(|X-500| < 100) \geq 1 - \frac{1}{100} = \frac{99}{100} = 0{,}99$.

Exercise 2: (12)

Suppose the joint density of two random variables X and Y is given by:

$$f(x,y) = \begin{cases} 6xy(2-x-y), & 0<x<1, 0<y<1 \\ 0, & otherwise, \end{cases} \qquad \text{(c)}$$

Compute the conditional expectation of X given that $Y = y$.

We need to Compute $\mathbb{E}[X|Y=y]$ for $0<y<1$.

First $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$ 02

But $f_Y(y) = \int_0^1 6xy(2-x-y)\,dx = 6y\left[x^2 - \frac{1}{3}x^3 - \frac{1}{2}x^2y\right]_0^1$

$f_Y(y) = 6y\left(1-\frac{1}{3}-\frac{1}{2}y\right) = y(4-3y).$ 03

Thus $f_{X|Y}(x|y) = \dfrac{6x(2-x-y)}{4-3y}$ 02

Hence $\mathbb{E}[X|Y=y] = \int_0^1 \dfrac{6x^2(2-x-y)}{4-3y}\,dx$

$= \dfrac{1}{4-3y}\left(\dfrac{5}{2}-2y\right).$

$= \dfrac{5-4y}{8-6y}.$ $\blacksquare$ 05

Exercise 3:(12)

The probability density function (pdf) of the duration of the (independent) interarrival times between successive cars on Dammam-Riyadh Highway is given by

$$f_T(t) = \begin{cases} \frac{1}{12} e^{-\frac{t}{12}}, & t \geq 0, \\ 0, & t < 0, \end{cases} \qquad \text{(d)}$$

where these durations are measured in seconds.

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A- An old fennec fox requires 12 seconds to cross the highway, and he starts out immediately after a car goes by. What is the probability that he will survive?

The arrival of cars is a Poisson process with arrival rate $\lambda = \frac{1}{12}$ cars/sec (since the interarrival times are iid exponential r.v.s with parameter $\frac{1}{12}$.) Therefore, the number $N(t)$ of arrival during a time interval of duration t, is a Poisson r.v with parameter $t/12$.
Hence this Fox will survive if and only if there is no cars during the 12 seconds it takes for him to cross the highway. Thus;
$P(\text{Fox survives}) = P(T > 12) = \int_{12}^{\infty} f_T(t)\,dt = e^{-\frac{12}{12}} = e^{-1}$.

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B- Another old fennec, slower but tougher, requires 24 seconds to cross the road, but it takes two cars to kill him. If he starts out at an arbitrary time, determine the probability that he survives.

The old Fennec will survive if and only if there are fewer than two cars during the 24 seconds it takes to cross, i.e the time of the second arrival, is greater than 24 seconds:

$P(\text{old Fennec survives}) = P(N(24) \leq 1) = P(N(24) = 0) + P(N(24)) = 1$
$= e^{-\frac{24}{2}} \frac{(24 \cdot \frac{1}{2})^0}{0!} + e^{-\frac{24}{2}} \frac{(24 \cdot \frac{1}{2})^1}{1!} = 3e^{-2}$

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C- If both these fennec foxes start out at the same time, immediately after a car goes by, what is the probability that exactly one of them survives?
(**Hint:** Consider a random variables N_1 = the number of cars in the first 12 seconds and N_2 = the number of cars in the second 12 seconds.)

$\{\text{Exactly one survives}\} = \{\text{only the } 1^{st} \text{ survives}\} \cup \{\text{only the second survives}\}$
$= \{N_1 = 0 \text{ and } N_2 > 1\} \cup \{N_1 = 1 \text{ and } N_2 = 0\}$.
N_1 and N_2 are independent, due to the memoryless property of the
Hence $P\{\text{Exactly one survives}\} = P(N_1=0)P(N_2>1) + P(N_1=1)P(N_2=0) = P_{N_1}(0)(1 - P_{N_2}(0) - P_{N_2}(1)) + P_{N_1}(1)P_{N_2}(0)$

Exercise 4: (15)

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with filtration $(\mathcal{F}_t)_{t\geq 0}$ and $B_t, t \geq 0$ be the standard Brownian motion with respect to P and $\mathcal{F}_t$.

Show that the process Y defined by $Y_t := t^2 B_t^3, t \geq 0$, satisfies the stochastic differential equation:

$$dY_t = \left(\frac{2}{t} Y_t + 3(t^4 Y_t)^{\frac{1}{3}}\right) dt + 3\,(t\,Y_t)^{\frac{2}{3}}\, dB_t,\ \ Y_0 = 0. \tag{e}$$

The function $f(t,x) := t^2 x^3$ is in C^2. 03

Applying Itô formula to f we get:

$$df(t,Y_t) = dY_t = \frac{\partial f}{\partial t}(t,B_t)\,dt + \frac{\partial f}{\partial x}(t,B_t)\,dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial^2 x}(t,B_t)\,(dB_t)^2$$

$$= 2tB_t^3\,dt + 3t^2B_t^2\,dB_t + \frac{1}{2}6t^2B_t\,dt \quad 03$$

$$= (2tB_t^3 + 3t^2B_t)\,dt + 3t^2B_t^2\,dB_t \quad 03$$

But $2tB_t^3 = \dfrac{2Y_t}{t}$; $3t^2B_t = 3(t^4 Y_t)^{\frac{1}{3}}$; $3t^2B_t^2 = 3(tY_t)^{\frac{2}{3}}$ 03

Hence $dY_t = \left(\frac{2}{t}Y_t + 3(t^4Y_t)^{\frac{1}{3}}\right)dt + 3(tY_t)^{\frac{2}{3}}\,dB_t.$ 03

Exercise 5:(13)

Let $B(t)$ be a 1-dimensional Brownian motion and $Y(t) = (Y_1(t), Y_2(t)) = (\cos(B(t)), \sin(B(t)))$.
Show that the Itô process $Y(t)$ can be written in the following form:

$$dY_t = a\, Y_t\, dt + K\, Y_t\, dB_t,$$

where a is a constant and K a suitable matrix.

- $dY_1(t) = -\sin B(t)\, dB_t - \frac{1}{2}\cos B(t)\, dt = -\frac{1}{2} Y_1\, dt - Y_2\, dB(t)$
- $dY_2(t) = \cos B(t)\, dB_t - \frac{1}{2}\sin B(t)\, dt = -\frac{1}{2} Y_2\, dt + Y_1\, dB(t)$

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Thus $dY_t = \begin{pmatrix} dY_1 \\ dY_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} Y_1 \\ -\frac{1}{2} Y_2 \end{pmatrix} dt + \begin{pmatrix} -Y_2 \\ Y_1 \end{pmatrix} dB(t).$ 05

Hence $dY_t = \overset{a}{\overset{\shortparallel}{-\frac{1}{2}}}\, Y(t)\, dt + K\, Y(t)\, dB(t)$, where 04

$K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; $a = -\frac{1}{2}$.

Exercise 6:(16)

The charge $Q(t)$ at time t at a fixed point in an electric circuit satisfies the 2−dimensional stochastic differential equation:

$$\begin{cases} LQ''_t + RQ'_T + \frac{1}{C}\,Q_t = G_t + \alpha\,W_t \\ Q(0) = Q_0, Q'(0) = I_0 \text{ given }, \end{cases} \tag{g}$$

where W_t is a one-dimensional white noise, L, R, C, α, I_0 are constants and G_t a given function.

1−Put

$$X_t(.) = \begin{pmatrix} X_1 = Q_t \\ X_2 = Q'_t \end{pmatrix} \tag{h}$$

and show that the SDE (g) can be written in the following form

$$dX_t = A\,X_t\,dt + H(t)\,dt + K\,dB_t, \tag{i}$$

where A, H and K are suitable matrices and B_t a Brownian motion. ; $W_t = \frac{dB_t}{dt}$,

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• $dX_t = \begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \begin{pmatrix} Q'_t \\ Q''_t \end{pmatrix} = \begin{pmatrix} X_2\,dt \\ -\frac{R}{L}X_2\,dt - \frac{1}{LC}X_1\,dt + \frac{1}{L}G_t\,dt + \frac{\alpha}{L}dB_t \end{pmatrix}$

Thus $dX_t = AX_t\,dt + H(t)dt + K\,dB_t$, where:

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$dX_t = \begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix}$; $A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix}$; $H(t) = \begin{pmatrix} 0 \\ \frac{1}{L}G_t \end{pmatrix}$; $K = \begin{pmatrix} 0 \\ \frac{\alpha}{L} \end{pmatrix}$

and B_t is a 1-dimensional Brownian motion.

2−Find the solution X_t of the stochastic differential equation (i)

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$e^{-At}dX_t - e^{-At}AX_t\,dt \overset{(*)}{=} e^{-At}[H(t)dt + K\,dB_t]$. Apply Itô Formula to $g(t, x_1, x_2) = e^{-At}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ gives: $d(e^{-At}X_t) = (-A)e^{-At}X(t)dt + e^{-At}dX_t + 0$

Substituting in $(*)$ gives: $e^{-At}X_t - X_0 = \int_0^t e^{-As}H(s)ds + \int_0^t e^{-As}K\,dB_s$, or

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$\int_0^t e^{-As}K\,dB_s = e^{-At}KB_t + \int_0^t B_s A K e^{-As}ds$ (Integ. by Parts).

Hence $X_t = e^{At}\left[X_0 + e^{-At}KB_t + \int_0^t e^{-As}(H(s) + AKB_s)ds\right]$.

Exercise 7:(17)
To model the spot freight rate in shipping, J.Tvedt(1995) used the geometric mean reversion process X_t which is defined as the solution of the stochastic differential equation

$$dX_t = \kappa\,(\alpha - \log X_t)\,X_t\,dt + \sigma\,X_t\,dB_t;\quad X_0 = x > 0, \qquad (j)$$

where κ, α, σ and x are positive constants.

1- Use the substitution $Y_t = \log X_t$ to transform the equation (j) into a linear stochastic differential equation for Y_t.

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Let $Y_t = \log X_t$, then $dY_t = \dfrac{dX_t}{X_t} - \dfrac{(dX_t)^2}{2X_t^2} = K(\alpha - Y_t)dt + \sigma dB_t - \dfrac{\sigma^2 X_t^2\,dt}{2X_t^2}$

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Hence $dY_t = (K\alpha - \frac{1}{2}\sigma^2)dt - KY_t dt + \sigma dB_t$.

2- Solve the Linear SDE obtained in **1**). (**Hint:** Apply Itô formula to $e^{\kappa t}Y_t$).

Apply Itō Formula to $e^{Kt}Y_t$ gives;

$d(e^{Kt}Y_t) = KY_t e^{Kt}dt + e^{Kt}dY_t = e^{Kt}\left[(K\alpha - \frac{1}{2}\sigma^2)dt + \sigma dB_t\right]$, and

$e^{Kt}Y_t - Y_0 = (K\alpha - \frac{1}{2}\sigma^2)\dfrac{e^{Kt}-1}{K} + \sigma\displaystyle\int_0^t e^{Ks}dB_s$. Hence

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$Y_t = e^{-Kt}Y_0 + (K\alpha - \frac{1}{2}\sigma^2)\dfrac{1-e^{-Kt}}{K} + \sigma\displaystyle\int_0^t e^{-K(t-s)}dB_s$.

3- Find the solution X_t of the SDE (j).

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$X_t = \exp\left\{e^{-Kt}\log x + (\alpha - \frac{\sigma^2}{2K})(1-e^{-Kt}) + \sigma e^{-Kt}\displaystyle\int_0^t e^{Ks}dB_s\right\}$.