Key Solutions MATH 642

1) The characteristic polynomial of A is

$$P_A(\lambda) = \det\begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda)^3 \qquad 2$$

The minimal polynomial: The possible factors of $P_A(\lambda)$ are $(2-\lambda)$, $(2-\lambda)^2$ and $(2-\lambda)^3$. 2

We have

$$(2.I-\lambda)\Big|_{\lambda=A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 0 \qquad 3$$

Therefore $m_A(x) = 2-x$ 3

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2)(a) Stationary points

$$\begin{cases} 0 = x_2 \\ 0 = x_1 - \dfrac{x_2^4}{x_1^2} + \sqrt{3+1} \\ 0 = x_1^2 + 3^2 \end{cases} \quad 4 \qquad \Rightarrow (x_{10}, x_{20}, u_0) = (-2, 0, 3) \quad 3$$

$$y_0 = g(x_{10}, x_{20}, u_0) = 13 \qquad 3$$

where $g(x_1, x_2, u) = y = x_1^2 + u^2$

(b) Linearization around $(-2, 0, 3)$

$$\frac{\partial f_1}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_2} = 1, \quad \frac{\partial f_1}{\partial u} = 0$$

where $f_1(x_1, x_2, u)$ is the first equation

$$\frac{\partial f_2}{\partial x_1} = 1 + 2\frac{x_2^4}{x_1^3}, \quad \frac{\partial f_2}{\partial x_2} = -4\frac{x_2^3}{x_1^2}, \quad \frac{\partial f_2}{\partial u} = \frac{1}{2\sqrt{u+1}}$$

$$\frac{\partial g}{\partial x_1} = 2x_1, \quad \frac{\partial g}{\partial x_2} = 0, \quad \frac{\partial g}{\partial u} = 2u$$

or, estimated at $\{x_0, u_0\}$

$$\left.\frac{\partial f_1}{\partial x_1}\right|_{\{x_0,u_0\}} = 0, \quad \frac{\partial f_1}{\partial x_2} = 1, \quad \frac{\partial f_1}{\partial u} = 0$$

$$\frac{\partial f_2}{\partial x_1} = 1, \quad \frac{\partial f_2}{\partial x_2} = 0, \quad \frac{\partial f_2}{\partial u} = \frac{1}{4}$$

$$\frac{\partial g}{\partial x_1} = -4, \quad \frac{\partial g}{\partial x_2} = 0, \quad \frac{\partial g}{\partial u} = 6$$ 3

So $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $B = \begin{pmatrix} 0 \\ 1/4 \end{pmatrix}$; $C = \begin{pmatrix} -4 & 0 \end{pmatrix}$ $D = (6)$

Let $\Delta x_1 = x_1 - x_{10}$, $\Delta x_2 = x_2 - x_{20}$ 1

$\Delta u = u - u_0$, $\Delta y = y - y_0$

$$\begin{pmatrix} \frac{\Delta x_1}{dt} \\ \frac{\Delta x_2}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/4 \end{pmatrix} u$$ 3

$$\Delta y = \begin{pmatrix} -4 & 0 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} + (6) u$$ 3

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3) The transfer-matrix is given by

$$\hat{G}(s) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \left(sI - \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 1$$

$$= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} s+2 & -1 & 0 \\ 0 & s & 0 \\ 0 & 0 & s+1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 1$$

To compute $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} s+2 & -1 & 0 \\ 0 & s & 0 \\ 0 & 0 & s+1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ 5

we can solve

$$\begin{pmatrix} s+2 & -1 & 0 \\ 0 & s & 0 \\ 0 & 0 & s+1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \quad z_2 = 0, \; z_3 = \frac{-1}{s+1}, \quad z_1 = \frac{1}{s+2}$$ 5

and therefore

$$\hat{G}(s) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{s+2} \\ 0 \\ \frac{-1}{s+1} \end{pmatrix} + 1 = \frac{s^2+3s+1}{(s+1)(s+2)}$$ 5

The system is BIBO stable since all poles have strictly negative real parts. 5

4) (a) We know that $\mathcal{L}\{e^{At}\} = (sI - A)^{-1}$

$$\mathcal{L}\{e^{At}\} = \begin{pmatrix} \frac{1}{s^2}+\frac{1}{s} & \frac{1}{s^2} & \frac{1}{s^2} \\ H_1(s) & \frac{1}{s^2}+\frac{1}{s} & \frac{1}{s^2} \\ H_2(s) & H_3(s) & \frac{-2}{s^2}+\frac{1}{s} \end{pmatrix} \Rightarrow \quad 3$$

$$\begin{pmatrix} \frac{1}{s^2}+\frac{1}{s} & \frac{1}{s^2} & \frac{1}{s^2} \\ H_1(s) & \frac{1}{s^2}+\frac{1}{s} & \frac{1}{s^2} \\ H_2(s) & H_3(s) & \frac{-2}{s^2}+\frac{1}{s} \end{pmatrix} \begin{pmatrix} s-1 & -1 & -1 \\ -1 & s-1 & -1 \\ 2 & 2 & 2+s \end{pmatrix} = I_d \quad 3$$

$$(s-1)H_1(s) - \frac{1}{s^2} - \frac{1}{s} + \frac{2}{s^2} = 0 \Rightarrow H_1(s) = \frac{1}{s^2} \to h_1(t) = t$$

$$\left.\begin{aligned} &(s-1)H_2(s) - H_3(s) - \frac{4}{s^2} + \frac{2}{s} = 0 \\ &(s-1)H_2(s) - H_3(s) - \frac{4}{s^2} + \frac{2}{s} = 0 \\ &-H_2(s) - H_3(s) - \frac{4}{s^2} + \frac{2}{s} - \frac{2}{s} + 1 \end{aligned}\right\} \Rightarrow \begin{aligned} &H_2(s) = H_3(s) = \frac{-2}{s^2} \\ &h_2(t) = h_3(t) = -2t \end{aligned}$$

4

(b) Since $\Phi(t_0, t_0) = I$ and $\frac{d}{dt}\Phi(t, t_0) = A(t)\Phi(t, t_0)$,
for $t = t_0$, we have $\left.\frac{d}{dt}\Phi(t, t_0)\right|_{t=t_0} = A(t_0)$. 5

For the state transition matrix, it is clear that

$$\frac{d}{dt}\Phi(t, t_0) = \begin{pmatrix} 0 & t e^{\frac{t^2-t_0^2}{2}} \\ 0 & t e^{\frac{t^2-t_0^2}{2}} \end{pmatrix} \Rightarrow A(t_0) = \begin{pmatrix} 0 & t_0 \\ 0 & t_0 \end{pmatrix}, \ \forall t_0.$$

5

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5) (a) The matrix has eigenvalues/eigenvectors

$$\lambda_1 = 2, \quad (A-2I)v = 0 \Leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ 2

$$\Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \forall \alpha \in \mathbb{R}^*$$

$$\lambda_2 = -2, \quad (A+2I)v = 0 \Leftrightarrow \begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \quad \forall \alpha \in \mathbb{R}^*$$ 2

and therefore can be diagonalized as

$$A = TDT^{-1}, \quad D = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}; \quad T = \begin{pmatrix} 1 & 1 \\ 0 & -4 \end{pmatrix}$$

and $T^{-1} \begin{pmatrix} 1 & 1/4 \\ 0 & -1/4 \end{pmatrix} \Rightarrow e^{tA} = Te^{tD}T^{-1}$

$$= \begin{pmatrix} 1 & 1 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 0 & 1/4 \\ 0 & -1/4 \end{pmatrix} = \begin{pmatrix} e^{2t} & \frac{e^{2t}-e^{-2t}}{4} \\ 0 & e^{-2t} \end{pmatrix}.$$ 2

Alternatively, one can compute e^{tA} by

$$e^{tA} = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right] = \mathcal{L}^{-1}\left\{\begin{pmatrix} s-2 & -1 \\ 0 & s+2 \end{pmatrix}^{-1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\begin{pmatrix} \frac{1}{s-2} & \frac{1/4}{s-2} - \frac{1/4}{s+2} \\ 0 & \frac{1}{s+2} \end{pmatrix}\right\} = \begin{pmatrix} e^{2t} & \frac{e^{2t}-e^{-2t}}{4} \\ 0 & e^{-2t} \end{pmatrix}$$ 2

5) (a) The matrix has eigenvalues / eigenvectors

$\lambda_1 = 2, \quad (A - 2I)v = 0 \Leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ 2

$\Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \forall \alpha \in \mathbb{R}^*$

$\lambda_2 = -2, \quad (A + 2I) v = 0 \Rightarrow \begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \ \forall \alpha \in \mathbb{R}^*$ 2

and therefore can be diagonalized as

$A = TDT^{-1}, \ D = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}; \ T = \begin{pmatrix} 1 & 1 \\ 0 & -4 \end{pmatrix}$

and $T^{-1} \begin{pmatrix} 1 & 1/4 \\ 0 & -1/4 \end{pmatrix} \Rightarrow e^{tA} = T e^{tD} T^{-1}$

$= \begin{pmatrix} 1 & 1 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 0 & 1/4 \\ 0 & -1/4 \end{pmatrix} = \begin{pmatrix} e^{2t} & \frac{e^{2t} - e^{-2t}}{4} \\ 0 & e^{-2t} \end{pmatrix}$. 2

Alternatively, one can compute e^{tA} by

$e^{tA} = \mathcal{L}^{-1}[(sI - A)^{-1}] = \mathcal{L}^{-1}\left\{ \begin{pmatrix} s-2 & -1 \\ 0 & s+2 \end{pmatrix}^{-1} \right\}$

$= \mathcal{L}^{-1}\left\{ \begin{pmatrix} \frac{1}{s-2} & \frac{1/4}{s-2} - \frac{1/4}{s+2} \\ 0 & \frac{1}{s+2} \end{pmatrix} \right\} = \begin{pmatrix} e^{2t} & \frac{e^{2t} - e^{-2t}}{4} \\ 0 & e^{-2t} \end{pmatrix}$

2

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(b) The system is unstable because the matrix has one eigenvalue with a positive real part. 2

6) (a) We have

$$\hat{g}(s) = \begin{pmatrix} 0 & 1 \end{pmatrix} \left(sI - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s & -1 \\ k_1 & s+k_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{s(s+k_2)+k_1} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s+k_2 & 1 \\ -k_1 & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{s}{s(s+k_2)+k_1}$$ 6

(b) By inspection, the desired transfer function is obtained for $k_1 = k_2 = 1$. 4

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