STAT 416- Major Exam 1

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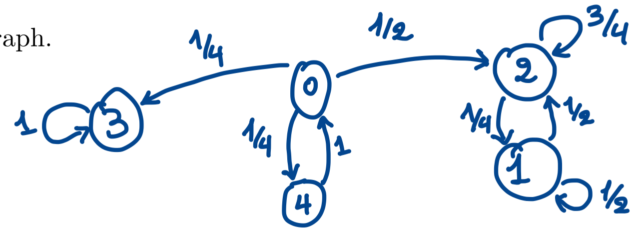
1 Exercise 1(20=3+4+3+5+5 points)

Consider the following Markov chain with states $\{0, 1, 2, 3, 4\}$ and transition probabilities matrix given by

$$P = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

1. Draw the transition graph.



2. Determine the classes, and specify which are positive recurrent, null recurrent or transient.

$C(0) = \{0, 4\}$: transient

$C(1) = \{1, 2\}$: positive recurrent

$C(3) = \{3\}$: positive recurrent

3. Find the period of all states.

$d(0) = d(4) = GCD\{2, 4, 6, \ldots\} = 2$

$d(1) = d(2) = GCD\{1, 2, \ldots\} = 1$

$d(3) = GCD\{1, \ldots\} = 1$

4. Find f_{01}. By one-step calculation, we have

$f_{01} = \frac{1}{2}\underbrace{f_{21}}_{1} + \frac{1}{4}\underbrace{f_{31}}_{0} + \frac{1}{4}f_{41} \Rightarrow f_{01} = \frac{1}{2} + \frac{1}{4}f_{41}$. Also, we have $f_{41} = 1 \times f_{01} = f_{01}$.

Then $f_{01} = \frac{1}{2} + \frac{1}{4}f_{01} \Rightarrow \frac{3}{4}f_{01} = \frac{1}{2} \Rightarrow f_{01} = \frac{4}{3} \times \frac{1}{2} = \frac{2}{3}$

5. Find $\lim_{n\to\infty} p_{01}^{(n)}$.

$$\begin{cases} (\pi_1, \pi_2)\begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} \\ \pi_1 + \pi_2 = 1 \end{cases} \Rightarrow \begin{cases} \frac{1}{2}\pi_1 + \frac{1}{4}\pi_2 = \pi_1 \\ \pi_1 + \pi_2 = 1 \end{cases} \Rightarrow \begin{cases} \pi_2 = 2\pi_1 \\ \pi_1 + 2\pi_1 = 1 \end{cases} \Rightarrow \pi_1 = 1/3$$

$$\lim_{n\to\infty} p_{01}^{(n)} = f_{01} \times \pi_1 = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}.$$

2 Exercise 2(15=4+6+5 points)

In a good weather year the number of storms is Poisson distributed with mean 1: in a bad year it is Poisson distributed with mean 3. Suppose that any year's weather conditions depends on past years only through the previous year's condition. Suppose that a good year is equally likely to be followed by a bad year as by a good year. A bad year is twice as likely to be followed by a bad year as by a good year. Suppose that last year -call it year 0- was a good year. Two states : G: good, B: bad.

1. Find the expected total number of storms in the next two years.

$$P=\begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} : P_{GG}\times 1+P_{GB}\times 3=\frac{1}{2}\times 1+\frac{1}{2}\times 3=2$$

$$P^2=\begin{bmatrix} 5/12 & 7/12 \\ 7/18 & 11/18 \end{bmatrix} : P_{GG}^{(2)}\times 1+P_{GB}^{(2)}\times 3=\frac{5}{12}\times 1+\frac{7}{12}\times 3=\frac{26}{12}=\frac{13}{6}$$

Total: $2+\frac{13}{6}=\frac{25}{6}$.

2. Find the probability there are no storms in year 3.

$$P^3=\begin{bmatrix} 29/72 & 43/72 \\ 43/108 & 65/108 \end{bmatrix}.$$

$$P_{GG}^{(3)}\times e^{-1}+P_{GB}^{(3)}\times e^{-3}=\frac{29e^{-1}+43e^{-3}}{72}$$

3. Find the long-run average number of storms per year.

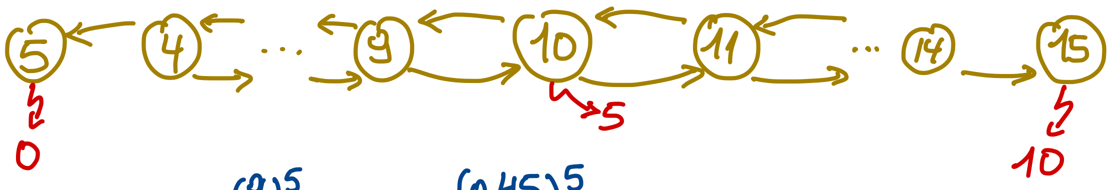
$$\begin{cases} (\pi_G,\pi_B)\begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}=(\pi_G,\pi_B) \\ \pi_G+\pi_B=1 \end{cases} \Rightarrow \begin{cases} \frac{1}{2}\pi_G+\frac{1}{3}\pi_B=\pi_G \\ \pi_G+\pi_B=1 \end{cases} \Rightarrow \begin{cases} \pi_B=\frac{3}{2}\pi_G \\ \pi_G+\frac{3}{2}\pi_G=1 \end{cases} \Rightarrow \begin{cases} \pi_B=\frac{3}{5} \\ \pi_G=\frac{2}{5} \end{cases}$$

Answer: $\pi_G\times 1+\pi_B\times 3=\frac{2}{5}+\frac{3}{5}\times 3=\frac{11}{5}$.

3 Exercise 3(15=8+7 points)

Ellen bought a share of stock for \$10, and it is believed that the stock price (day by day) will increase by \$1 with probability $p=0.55$ or decrease by \$1 with probability $q=0.45$.

1. What is the probability that Ellen's stock reaches the high value of \$15 before the low value of \$5?

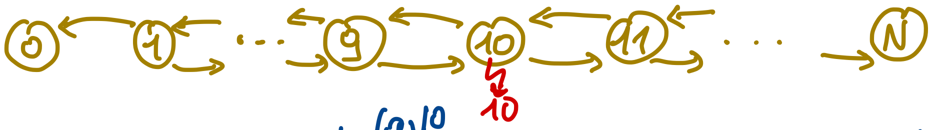


$$\begin{cases} i=5 \\ N=10 \end{cases}$$

$$u_5=\frac{1-\left(\frac{q}{p}\right)^5}{1-\left(\frac{q}{p}\right)^{10}}=\frac{1-\left(\frac{0.45}{0.55}\right)^5}{1-\left(\frac{0.45}{0.55}\right)^{10}}=0.7317$$

2. What is the probability that Ellen will become infinitely rich?

$$\left(\frac{q}{p}\right)^n \to 0$$



$$\lim_{N\to\infty} u_N = \lim_{N\to\infty} \frac{1-\left(\frac{q}{p}\right)^{10}}{1-\left(\frac{q}{p}\right)^N} = 1-\left(\frac{q}{p}\right)^{10} = 1-\left(\frac{0.45}{0.55}\right)^{10} = 0.8655$$

Hint: $U_i = \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N}$

4 Exercise 4(15=7+8 points)

The number of individuals produced by each individual in a branching process is 1 or 2 with the same probability p, and 0 with probability $1-2p$, for $0 < p < 1/2$, independently of all others.

1. Find the condition on p so that the extinction probability is less than 1.

$$\mu = 0\times(1-2p) + 1\times p + 2\times p = 3p > 1$$

$$\iff p > \frac{1}{3}$$

2. Find the extinction probability under this condition.

π_0 is the smallest solution in $(0,1)$ of the equation $\varphi(s)=s$

$$\Rightarrow 1-2p+ps+ps^2 = s \Rightarrow ps^2+(p-1)s+1-2p=0$$

As 1 is solution, then product $= 1\times\pi_0 = \dfrac{1-2p}{p} = \dfrac{1}{p}-2$

$$\Rightarrow \boxed{\pi_0 = \frac{1}{p}-2}$$

5 Exercise 5(15=8+7 points)

Consider a random walk on A, B, C, D, E, F given by Figure 1. For example if at time n we are at E, then at time $n+1$, we will be at A, C, D or F with probability 1/4

1. Find the proportion of long-term time that we are at F.

We have $d(A)=d(C)=3$, $d(B)=d(D)=d(F)=2$ and $d(E)=4$.

Then $\pi_F = \dfrac{d(F)}{\sum_i d(i)} = \dfrac{2}{2\times3+3\times2+4} = 1/8.$

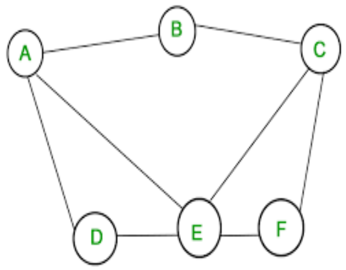


Figure 1: Random walk on graphs.

2. Find the number of steps expected to return the first time to A if we leave A.

$$m_A = \frac{1}{\pi_A} = \frac{\sum_i d(i)}{d(A)} = \frac{16}{3}$$

6 Exercise 6(10 points)

Four out of every six trucks on the road are followed by a car, while only two out every seven cars are followed by a truck. What fraction of vehicles on the road are trucks?

Vehicle n is truck (T) or car (C).

Then, the transition matrix is
$$\begin{array}{c c} & \begin{array}{cc} C & T \end{array} \\ \begin{array}{c} C \\ T \end{array} & \begin{bmatrix} 5/7 & 2/7 \\ 4/6 & 2/6 \end{bmatrix} \end{array}.$$

$$\Rightarrow \begin{cases} (\pi_C, \pi_T) \begin{bmatrix} 5/7 & 2/7 \\ 4/6 & 2/6 \end{bmatrix} = (\pi_C, \pi_T) \\ \pi_C + \pi_T = 1 \end{cases} \Rightarrow \begin{cases} \frac{5}{7}\pi_C + \frac{4}{6}\pi_T = \pi_C \\ \pi_C + \pi_T = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \pi_T = \frac{3}{7}\pi_C \\ \pi_C + \pi_T = 1 \end{cases} \Longrightarrow \pi_C + \frac{3}{7}\pi_C = 1 \Rightarrow \begin{cases} \pi_C = \frac{7}{10} \\ \pi_T = \frac{3}{10} \end{cases}$$

Answer $\pi_T = \frac{3}{10}$