STAT 416- Major Exam 2

KFUPM, Department of Mathematics and Statistics

Kroumi Dhaker, Term 221

Write all details and Justify your answers

1 Exercise 1(5+5+5 points)

Suppose that $N(t)$ is a Poisson process with rate 2. Compute

1. $P(N(3) = 4|N(1) = 1)$

$= \mathbb{P}\{N(3) - N(1) = 3 \mid N(1) = 1\}$

$= \mathbb{P}\{N(3) - N(1) = 3\}$ $\quad N(3) - N(1) \sim \mathcal{P}(2 \times (3-1)) = \mathcal{P}(4)$

$= e^{-4} \times \frac{4^3}{3!}$

2. $E(N(5)|N(2) = 5)$

$= E[N(5) - N(2) + N(2) \mid N(2) = 5]$

$= E[N(5) - N(2) \mid N(2) = 5] + E[N(2) \mid N(2) = 5]$

$= 6 + 5$ $\quad N(5) - N(2) \sim \mathcal{P}(2 \times (5-2)) = \mathcal{P}(6)$

$= 11$

3. $P(N(1) = 1|N(3) = 4)$

$$= \frac{\mathbb{P}\{N(1) = 1, N(3) = 4\}}{\mathbb{P}\{N(3) = 4\}} = \frac{\mathbb{P}\{\underbrace{N(1)}_{\sim \mathcal{P}(2)} = 1, \underbrace{N(3) - N(1)}_{\sim \mathcal{P}(4)} = 3\}}{\mathbb{P}\{\underbrace{N(3)}_{\mathcal{P}(6)} = 4\}}$$

$$= \frac{e^{-2} \times 2^1/1! \times e^{-4} \times 4^3/3!}{e^{-6} \times \frac{6^4}{4!}} = \frac{4!}{1! \times 3!} \times \left(\frac{2}{6}\right)^1 \times \left(\frac{4}{6}\right)^3$$

$$= \binom{4}{1} \left(\frac{2}{6}\right)^1 \times \left(\frac{4}{6}\right)^3$$

2 Exercise 2(8+4 points)

An insurance company pays out claims at times of a Poisson process with rate 4 per week. Suppose that the mean payment is 400 and the standard deviation if 200. Find the mean and standard deviation of the total payments for 6 weeks.

Let Y be the amount per claim.

We have $E[Y] = 400$,

$$E[Y^2] = Var(Y) + E[Y]^2 = 200^2 + 400^2$$

$$= 40,000 + 160,000 = 200,000.$$

Let $N(t)$ be the number of claims in $[0,t]$. $N(t)$ is a Poisson process with rate $\lambda = 4$. The total payments in 6 weeks is

$$X(6) = \sum_{i=1}^{N(6)} Y_i.$$

We have $E[X(6)] = E[Y] \times E[N(6)] = 400 \times 4 \times 6 = 9,600\$$

and $\sigma(X(6)) = \sqrt{E[Y^2] \times E[N(6)]}$

$$= \sqrt{200,000 \times 4 \times 6}$$

$$= \sqrt{4,800,000}$$

3 Exercise 3(17=5+6+6 points)

Consider a two-server system in which a customer is served first by server 1, then by server 2, and then departs. The service times at server i are exponential random variables with rates μ_i, for $i = 1, 2$. When you arrive, you find server 1 free and two customers at server 2: customer A in service and customer B waiting in line.

1. Find P_A, the probability that A is still in service when you move over to server 2.

$P_A = \frac{\mu_1}{\mu_1+\mu_2}$. This is the probability that you will finish service with server 2 before A finishes.

A → You

You

$\mu_1 * \mu_2$

$\frac{\mu_2}{(\mu_1+\mu_2)}$

2. Find P_B, the probability that B is in service when you move over to server 2.

$$P_B = \underbrace{\frac{\mu_2}{\mu_1+\mu_2}}_{\text{A finishes first}} * \frac{\mu_1}{\mu_1+\mu_2} \rightarrow \text{you finish before B}$$

3. Find $E[T]$, where T is the time that you spend in the system.

$$E[T] = \underbrace{\frac{1}{\mu_1}}_{\text{Your service time with server 1}} + \underbrace{\frac{1}{\mu_2}}_{\text{Your service time with server 2}} + P_A * \underbrace{\frac{2}{\mu_2}}_{\text{your waiting time (A and B will be served)}} + P_B * \frac{1}{\mu_2} \rightarrow \text{your waiting time (B will be served)}$$

You finish before A

A finish the first and then y

$$= \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{\mu_1}{\mu_1+\mu_2} * \frac{2}{\mu_2} + \frac{\mu_1\mu_2}{(\mu_1+\mu_2)^2} * \frac{1}{\mu_2} = \frac{3\mu_1^2 + 6\mu_1^2\mu_2 + 3\mu_1\mu_2^2 + \mu_2^3}{\mu_1\mu_2(\mu_1+\mu_2)^2}$$

Other way

$$E[T] = \underbrace{\frac{1}{\mu_1+\mu_2}}_{\substack{\text{time to the first}\\ \text{finisher among}\\ \text{you and A}}} + \overbrace{\frac{\mu_1}{\mu_1+\mu_2}}^{\text{you finish before A}} \times \left[\underbrace{\frac{1}{\mu_2}}_{A} + \underbrace{\frac{1}{\mu_2}}_{B} + \underbrace{\frac{1}{\mu_2}}_{\text{You}} \right] + \overbrace{\frac{\mu_2}{\mu_1+\mu_2}}^{\substack{\text{A finishes}\\ \text{before you}}} \times \Bigg[\overbrace{\frac{1}{\mu_1+\mu_2}}^{\substack{\text{time to the first}\\ \text{finisher among}\\ \text{you and B}}} + \underbrace{\frac{\mu_1}{\mu_1+\mu_2}}_{\substack{\text{You finish}\\ \text{before B}}} \times \left(\underbrace{\frac{1}{\mu_2}}_{B} + \underbrace{\frac{1}{\mu_2}}_{\text{you}} \right)$$

$$+ \underbrace{\frac{\mu_2}{\mu_1+\mu_2}}_{\substack{\text{B finishes}\\ \text{before you}}} \times \left(\underbrace{\frac{1}{\mu_1}}_{\substack{\text{your service}\\ \text{time with}\\ \text{server 1}}} + \underbrace{\frac{1}{\mu_2}}_{\substack{\text{your service}\\ \text{time with}\\ \text{server 2}}} \right) \Bigg]$$

$$= \frac{1}{\mu_1+\mu_2} + \frac{3\mu_1}{(\mu_1+\mu_2)\mu_2} + \frac{\mu_2}{\mu_1+\mu_2} \times \left[\frac{1}{\mu_1+\mu_2} + \frac{2\mu_1}{(\mu_1+\mu_2)\mu_2} + \frac{\cancel{\mu_2}}{\cancel{\mu_1+\mu_2}} \times \frac{\cancel{\mu_1+\mu_2}}{\mu_1\cancel{\mu_2}} \right]$$

$$= \frac{1}{\mu_1+\mu_2} + \frac{3\mu_1}{(\mu_1+\mu_2)\mu_2} + \frac{\mu_2}{\mu_1+\mu_2} \times \Bigg[\underbrace{\frac{1}{\mu_1+\mu_2} + \frac{2\mu_1}{(\mu_1+\mu_2)\mu_2} + \frac{1}{\mu_1}}_{\frac{\mu_1\mu_2 + 2\mu_1^2 + (\mu_1+\mu_2)\mu_2}{\mu_1\mu_2(\mu_1+\mu_2)}} \Bigg]$$

$$= \frac{3\mu_1^3 + 6\mu_1^2\mu_2 + 3\mu_1\mu_2^2 + \mu_2^3}{\mu_1\mu_2(\mu_1+\mu_2)^2}$$

4 Exercise 4(16=4+4+4+4 points)

A service center consists of two servers, each working at an exponential rate of two services per hour. If customers arrive at a Poisson rate of three per hour, then, assuming a system capacity at a most three customers. Let X_t be the number of customers in the system at time $t \geq 0$.

1. Determine the generator of $(X_t)_t$.

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} -3 & 3 & 0 & 0 \\ 2 & -5 & 3 & 0 \\ 0 & 4 & -7 & 3 \\ 0 & 0 & 4 & -4 \end{bmatrix} \end{array}$$

2. Find the stationary distribution associated to $(X_t)_t$.

$\theta_0 = 1$; $\theta_1 = \frac{3}{2}$; $\theta_2 = \frac{3 \times 3}{2 \times 4} = \frac{9}{8}$; $\theta_3 = \frac{3 \times 3 \times 3}{2 \times 4 \times 4} = \frac{27}{32}$

$\sum_{n=0}^{3} \theta_n = 1 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32} = \frac{143}{32}$. Here, we have $\pi_K = \frac{\theta_K}{\sum \theta_n}$.

More precisely, we have $\pi_0 = \frac{32}{143}$, $\pi_1 = \frac{43}{143}$, $\pi_2 = \frac{36}{143}$ and

$\pi_3 = \frac{27}{143}$.

3. What fraction of potential customers enter the system.

$$1 \times \pi_0 + 1 \times \pi_1 + 1 \times \pi_2 + \underbrace{0 \times \pi_3}_{} = \frac{116}{143}.$$

If we have three customers in the system, the probability that a new customer enters the system is 0.

4. What is the average number of customers in the system.

$$0 \times \pi_0 + 1 \times \pi_1 + 2 \times \pi_2 + 3 \times \pi_3 = \frac{196}{143}.$$

5 Exercise 5(20=4+4+4+8 points)

There are two machines, one of which is used as a spare. A working machine will function for an exponential time with rate λ and will then fail. Upon failure, it is immediately replaced by the other machine if that one is in working order, and it goes to the repair facility. The repair facility consists of a single person who takes an exponential time with rate μ to repair a failed machine. At the repair facility, the newly failed machine enters service if the repairperson is free. If the repairperson is busy, it waits until the other machine is fixed; at that time, the newly repaired machine is put in service and repair begins on the other one. Let X_t be the number of down machines at time $t \geq 0$.

1. Give the birth and death rates.

$$\lambda_0 = \lambda_1 = \lambda$$

$$\mu_1 = \mu_2 = \mu$$

$$\begin{tikzcd} 0 \arrow[r, bend right, "\lambda"'] & 1 \arrow[l, bend right, "\mu"'] \arrow[r, bend right, "\lambda"'] & 2 \arrow[l, bend right, "\mu"'] \end{tikzcd}$$

2. Find the stationary distribution. Note that $\theta_0 = 1$ $\theta_1 = \frac{\lambda}{\mu}$ and $\theta_2 = \left(\frac{\lambda}{\mu}\right)^2$.

Then, we have $\theta_0 + \theta_1 + \theta_2 = 1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2} = \frac{\mu^2 + \lambda\mu + \lambda^2}{\mu^2}$, from which we get $\pi_0 = \theta_0 / \Sigma\theta_n = \frac{\mu^2}{\mu^2 + \lambda\mu + \lambda^2}$, $\pi_1 = \theta_1 / \Sigma\theta_n = \frac{\lambda\mu}{\mu^2 + \lambda\mu + \lambda^2}$ and $\pi_2 = \theta_2 / \Sigma\theta_n = \frac{\lambda^2}{\mu^2 + \lambda\mu + \lambda^2}$.

3. In the long run, what proportion of time is there a working machine?

$$1 - \pi_2 = \frac{\mu^2 + \lambda\mu}{\mu^2 + \lambda\mu + \lambda^2}.$$

4. Starting with both machines in working condition, find the expected value of the time until both are in the repair facility

Let T_i be the amount of time to move the first time to $i+1$ given that the system is now at i, for $i=0,1$.

$$E[T_0] = \frac{1}{\lambda}$$

$$E[T_1] = \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} * 0 + \frac{\mu}{\lambda+\mu} * \left[E[T_0]+E[T_1]\right]$$

$$\Rightarrow \frac{\lambda}{\lambda+\mu} E[T_1] = \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} * \frac{1}{\lambda} \Rightarrow E[T_1] = \frac{1}{\lambda} + \frac{\mu}{\lambda^2}$$

Answer: $E[T_0] + E[T_1] = \frac{2}{\lambda} + \frac{\mu}{\lambda^2}$.

6 Exercise 6(20=5+5+6+4 points)

Consider a taxi station where taxis and customers arrive in accordance with Poisson processes with respective rates of one and two per minute. A taxi will wait no matter how many other taxis are present. However, an arriving customer that does not find a taxi waiting leaves. Let X_t be the number of taxis waiting.

1. Specify the type of queueing system and find the corresponding parameters.

It is M|M|1, where

$\lambda_n = 1$ (a new taxi will arrive)

$\mu_n = 2$ (a customer will arrive and then a taxi will leave)

2. Show that the stationary distribution exists and find it.

$$\theta_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_n}{\mu_1 \mu_2 \cdots \mu_n} = \left(\frac{1}{2}\right)^n$$

It is clear that $\sum_n \theta_n = \sum \left(\frac{1}{2}\right)^n$ converges as it is a geometric series with raison $\left|\frac{1}{2}\right| < 1$. Then, the stationary distribution exists and we have

$$\pi_n = \frac{\theta_n}{\sum \theta_K} = \frac{(1/2)^n}{\frac{1}{1-\frac{1}{2}}} = \frac{1}{2} * \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{n+1}$$

3. Find the average number of taxis waiting.

$$\sum_{n=0}^{\infty} n\pi_n = \sum_{n=1}^{\infty} n \times \frac{1}{2} \times \left(\frac{1}{2}\right)^n$$

$$= \frac{1}{2} \times \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = \frac{1}{2} \times \frac{1/2}{1/4} = 1$$

4. The proportion of arriving customers that get taxis.

Any customer who arrives and finds at least a taxi waiting will get taxi. Then, the answer is

$$1 - \pi_0 = 1 - \left(\frac{1}{2}\right)^{0+1} = \frac{1}{2}.$$

Hint:

$$\sum_{n=m}^{\infty} x^n = \frac{x^m}{1-x},$$

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}.$$