STAT 416- Final Exam

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Write all details and Justify your answers

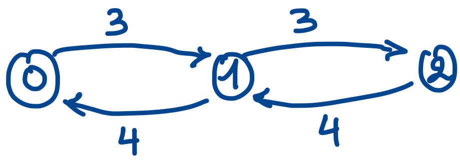
Exercise 1 (15 points)

A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variables with mean $\frac{1}{4}$.

1. What is the average number of customers in the shop?
2. What fraction of potential customers are lost?
3. If the barber could work twice as fast, how more business would he do?

Let X_t be the number of customers in the barbershop at time $t \geq 0$

$(X_t)_{t\geq 0}$ is a birth-death process with $\lambda_i = 3$ and $\mu_i = 4$



$$\theta_0 = 1; \quad \theta_1 = \frac{\lambda_0}{\mu_1} = \frac{3}{4} \text{ and } \theta_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

$$\theta_0 + \theta_1 + \theta_2 = 1 + \frac{3}{4} + \frac{9}{16} = \frac{16+12+9}{16} = \frac{37}{16}$$

$$\Rightarrow \quad \pi_0 = \frac{\theta_0}{\theta_0 + \theta_1 + \theta_2} = \frac{1}{\frac{37}{16}} = \frac{16}{37}$$

$$\pi_1 = \frac{\theta_1}{\theta_0 + \theta_1 + \theta_2} = \frac{3/4}{\frac{37}{16}} = \frac{12}{37}$$

$$\pi_2 = \frac{\theta_2}{\theta_0 + \theta_1 + \theta_2} = \frac{\frac{9}{16}}{\frac{37}{16}} = \frac{9}{37}$$

1/ $0 \times \pi_0 + 1 \times \pi_1 + 2 \times \pi_2 = \frac{12}{37} + \frac{18}{37} = \frac{30}{37}$

2/ $\pi_2 = \frac{9}{37}$

3/ $\mu = 8 \Rightarrow \theta_0 = 1,\ \theta_1 = 3/8$ and $\theta_2 = \left(\frac{3}{8}\right)^2 = \frac{9}{64}$

$$\Rightarrow \theta_0 + \theta_1 + \theta_2 = 1 + \frac{3}{8} + \frac{9}{64} = \frac{97}{64}$$

$$\Rightarrow \pi_0 = \frac{1}{\frac{97}{64}} = \frac{64}{97},\quad \pi_1 = \frac{\frac{3}{8}}{\frac{97}{64}} = \frac{24}{97} \text{ and } \pi_2 = \frac{\frac{9}{64}}{\frac{97}{64}} = \frac{9}{97}.$$

Now, the proportion of lost customers is $\pi_2 = \frac{9}{97}$.

So the proportion of lost customers passes from $\frac{9}{64}$ to $\frac{9}{97}$.

Exercise 2 (15 points)

The manager of a market ca hire either Mary or Alice. Mary, who gives service at exponential rate of 20 customers per hour, can be hired at a rate of \$3 per hour. Alice, who gives service at exponential rate of 30 customers per hour, can be hired at a rate \$$C$ per hour. The manager estimates that, on the average, each customer's time is worth \$1 per hour and should be accounted for in the model. Assume customers arrive at a Poisson rate of 10 per hour.

1. What is the average cost per hour if Mary is hired? If Alice is hired?

2. Find C is the average cost per hour is the same for Mary and Alice.

a/ Mary: It is an M|M|1 system with $\lambda = 10$ and $\mu = 20$.
So in average, we have $\frac{\lambda}{\mu - \lambda} = \frac{10}{20-10} = 1$ customer per hour in the system. The cost for the market is

$$1 \times 1\$ + 3\$ = 4\$ \text{ per hour}$$

Alice: It is an M|M|1 system with $\lambda = 10$ and $\mu = 30$.
So in average, we have $\frac{\lambda}{\mu - \lambda} = \frac{10}{30-10} = \frac{1}{2}$ customer per hour in the system. The cost for the market is

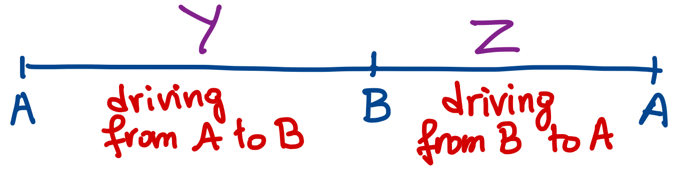
$$\frac{1}{2} \times 1\$ + C = C + \frac{1}{2} \text{ per hour}$$

b/ The cost is the same if $4 = C + \frac{1}{2} \Rightarrow C = \frac{7}{2}$.

Exercise 3 (15 points)

A truck driver regularly drives round trips from A to B and then back to A. Each time he drives from A to B, he drives at a fixed speed that (in miles per hour) is uniformly distributed between 40 and 60. Each time he drives from B to A, he drives at a fixed speed that is equally likely to be either 40 or 60.

1. In the long run, what proportion of his driving time is spent going from B to A?
2. In the long run, for what proportion of his driving time is he driving at a speed of 40 miles per hour?



alternating renewal process

A renewal corresponds to "arrive to A"

Let $T_{i,j}$ be the time to go from i to j and S be the speed when the driver is going from A to B.

Then, we have $E[T_{A,B}] = E[E[T_{A,B} | S]] = \int_{40}^{60} E[T_{A,B} | S=s] \frac{ds}{20}$

$$= \int_{40}^{60} \frac{x}{s} \cdot \frac{ds}{20} = \frac{x}{20} [\ln s]_{40}^{60} = \frac{x}{20} \ln\left(\frac{3}{2}\right),$$

where x is the distance from A to B.

In the other hand, we have $E[T_{B,A}] = \frac{1}{2} \times \frac{x}{40} + \frac{1}{2} \times \frac{x}{60} = \frac{x}{48}$

drive at speed 40

time to go from B to A

drive at speed 60

time to go from B to A

1/ By the alternating renewal process's theorem, the proportion of the driving time spent going to B is

$$\frac{E[T_{B,A}]}{E[T_{A,B}] + E[T_{B,A}]} = \frac{\frac{x}{48}}{\frac{x}{20}\ln\left(\frac{3}{2}\right) + \frac{x}{48}} = \frac{1}{1 + \frac{48}{20}\ln\left(\frac{3}{2}\right)}$$

2/ The long-run proportion of driving time at a speed of 40 m per hour is

$$\frac{\frac{1}{2} \times \frac{x}{40}}{E[T_{A,B}] + E[T_{B,A}]} = \frac{\frac{1}{2} \times \frac{x}{40}}{\frac{x}{20}\ln\left(\frac{3}{2}\right) + \frac{x}{48}} = \frac{1}{4\ln\left(\frac{3}{2}\right) + \frac{80}{48}}.$$

Exercise 4 (20 points)

A facility produces items according to a Poisson process with rate λ. However, it has shelf space for only k items and so it shuts down production whenever K items present. Customers arrive at the facility according to a Poisson process with rate μ. Each customer wants one item and will immediately depart either with the item or empty handed if there is no item available. Let X_t be the number of items in stock at time $t \geq 0$.

1. Show that $(X_t)_t$ is a birth-death process and give the birth and death rates.
2. Find its stationary distribution.
3. Find the proportion of customers that go away empty handed.
4. Find the average number of items in stock.
5. Find the average time that an item is on the shelf.

Hint: $\sum_{i=n}^{m} x^i = \frac{x^n - x^{m+1}}{1-x}$

1/ Number of items increases by one if the facility produces an item ($\lambda_i = \lambda$ for $i = 0, 1, \dots, K-1$) and decreases by one if a customer arrives to the facility ($\mu_i = \mu$ for $i = 1, 2, \dots, K$)

2/ We have $\theta_i = \left(\frac{\lambda}{\mu}\right)^i$ for $i = 0, 1, \dots, K$. Then,

$$\sum_{i=0}^{K} \theta_i = \sum_{i=0}^{K} \left(\frac{\lambda}{\mu}\right)^i = \frac{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}}{1 - \frac{\lambda}{\mu}}$$

$$\Rightarrow \pi_n = \frac{\theta_n}{\sum_{i=0}^{K} \theta_i} = \left(1 - \frac{\lambda}{\mu}\right) \times \frac{(\lambda/\mu)^n}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}} \quad \text{for } n = 0, 1, \dots, K$$

3/ The answer is $\pi_0 = \dfrac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}}$

4/ The average number is $\displaystyle\sum_{n=0}^{K} n\pi_n = \frac{\left(1 - \frac{\lambda}{\mu}\right) \sum_{n=1}^{K} n \left(\frac{\lambda}{\mu}\right)^n}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}} = L$

5/ $W = \dfrac{L}{\lambda_a} = \dfrac{\left(1 - \frac{\lambda}{\mu}\right) \sum_{n=1}^{K} n \left(\frac{\lambda}{\mu}\right)^n}{\lambda_a \left(1 - \left(\frac{\lambda}{\mu}\right)^{K+1}\right)}$, where $\lambda_a = \lambda \left[1 - \pi_K\right]$

Exercise 5 (15 points)

Jacob's car buying policy is to always buy a new car, repair all breakdowns that occur during the first T time units of ownership, and then junk the car and buy a new one at the first breakdown that occurs after the car has reached age T. Suppose that the time until the first breakdown of a new car is exponential with rate λ, and that each time a car is repaired the time until the next breakdown is exponential with rate μ.

1. At what rate does Jacob buy new cars?
2. Supposing that a new car costs C and that a cost r is incurred at each repair, what is Jacob's long run average cost per unit time.

a/ A renewal is when Jack buy a new car. The answer is $\frac{1}{E[X]}$, where X is the interarrival. Here, to find an expression of X we will use N: the number of breakdowns in $[0,T]$. More precisely, we have

$$E[X|N] = \begin{cases} T + E[Exp(\lambda)] = T + \frac{1}{\lambda} & \text{if } N=0 \\ T + E[Exp(\mu)] = T + \frac{1}{\mu} & \text{if } N \geq 1 \end{cases}$$

$$\Rightarrow E[X] = \left(T+\frac{1}{\lambda}\right) \underbrace{\mathbb{P}\{Exp(\lambda) > T\}}_{\int_T^\infty \lambda e^{-\lambda t} dt = e^{-\lambda T}} + \left(T+\frac{1}{\mu}\right) \cdot \mathbb{P}\{Exp(\lambda) < T\}$$

$$\Rightarrow E[X] = T + \frac{1}{\lambda} e^{-\lambda T} + \frac{1-e^{-\lambda T}}{\mu}$$

By the elementary renewal theorem, the rate is

$$\frac{1}{T + \frac{1}{\lambda} e^{-\lambda T} + \frac{1-e^{-\lambda T}}{\mu}}$$

b/ Let Y be the time to the first breakdown. If $Y \geq T$, then the total cost R is only C. If $Y < T$, then the number of breakdowns in the interval $[Y, T]$ is a Poisson process with rate $\mu \Rightarrow$ in average $\mu(T-Y)$

$$\Rightarrow E[R] = E[E[R|Y]] = \int_0^T \left(C + r + \mu(T-y)\right) \lambda e^{-\lambda y} dy + \int_T^\infty C \lambda e^{-\lambda y} dy$$

*) By two integrations by parts

$C+r+r\mu(T-y)$	$\lambda e^{-\lambda y}$
$-r\mu$	$-e^{-\lambda y}$
0	$\frac{1}{\lambda}e^{-\lambda y}$

, we get

$$\int_0^T \big(C+r+\mu(T-y)\big)\lambda e^{-\lambda y}\,dy = \Big[-\big(C+r+r\mu(T-y)\big)e^{-\lambda y}+\frac{r\mu}{\lambda}e^{-\lambda y}\Big]_{y=0}^{y=T}$$

$$= \Big(\frac{r\mu}{\lambda}-C-r\Big)e^{-\lambda T}+C+r+r\mu T-\frac{r\mu}{\lambda} \quad (1)$$

*) $$\int_T^{\infty} C\lambda e^{-\lambda y}\,dy = C\,e^{-\lambda T} \quad (2)$$

$$\Rightarrow E[R] = (1)+(2) = \Big(\frac{r\mu}{\lambda}-r\Big)e^{-\lambda T}+C+r\Big(1+\mu T-\frac{\mu}{\lambda}\Big)$$

Then, the average cost per unit time is

$$\frac{E[R]}{E[X]} = \frac{\Big(\frac{r\mu}{\lambda}-r\Big)e^{-\lambda T}+C+r\Big(1+\mu T-\frac{\mu}{\lambda}\Big)}{T+\frac{1}{\lambda}e^{-\lambda T}+\frac{1-e^{-\lambda T}}{\mu}}.$$

Exercise 6 (20 points)

Consider a Network of three stations with a single server at each station. Customers arrive at stations 1, 2, 3 in accordance with Poisson processes having respective rates 5, 10 and 15. The service times at the three stations are exponential with respective rates 10, 50 and 100.

- A customer completing service at station 1 is equally likely to (i) go to station 2, (ii) go to station 3, or (iii) leave the system.
- A customer departing service at station 2 always goes to station 3.
- A departure from service at station 3 is equally likely to either go to station 2 or leave the system.

1. Find the stationary distribution of the system state.
2. What is the average number of customers in the system?
3. What is the average time a customer spends in the system?

$P_{12} = P_{13} = 1/3 \;;\; P_{23} = 1 \;;\; P_{32} = 1/2$

$r_1 = \lambda_1 = 5$

$r_2 = \lambda_2 + P_{12} r_1 + P_{32} r_3 = 10 + \frac{5}{3} + \frac{r_3}{2} = \frac{35}{3} + \frac{r_3}{2}$ (1)

$r_3 = \lambda_3 + P_{13} r_1 + P_{23} r_2 = 15 + \frac{5}{3} + r_2 = \frac{50}{3} + r_2$ (2)

Inserting (2) in (1) leads to

$r_2 = \frac{35}{3} + \frac{1}{2}\left[\frac{50}{3} + r_2\right] \Rightarrow \frac{1}{2} r_2 = 20 \Rightarrow r_2 = 40$

(2) $\Rightarrow r_3 = \frac{50}{3} + 40 = \frac{170}{3}$.

1/ $\pi(n_1, n_2, n_3) = \prod_{i=1}^{3} \left(1 - \frac{r_i}{\mu_i}\right)\left(\frac{r_i}{\mu_i}\right)^{n_i}$

$= \left(1 - \frac{5}{10}\right) \times \left(\frac{5}{10}\right)^{n_1} \times \left(1 - \frac{40}{50}\right) \times \left(\frac{40}{50}\right)^{n_2} \times \left(1 - \frac{170}{300}\right) \times \left(\frac{170}{300}\right)^{n_3},$

for any $n_1, n_2, n_3 \geq 0$

2/ $$L = \sum_{i=1}^{3} \frac{r_i}{\mu_i - r_i} = \frac{5}{10-5} + \frac{40}{50-40} + \frac{170/3}{100-170/3}$$

$$= \frac{82}{13} \text{ customers}$$

3/ $$\lambda_a = r_1 + r_2 + r_3 = 5 + 40 + \frac{170}{3} = \frac{305}{3}$$

$$\Rightarrow W = \frac{L}{\lambda_a} = \frac{246}{3965}$$