STAT 416- Final Exam

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Instructions: You must show all your work. No materials are allowed.

1 Exercise (12 points)

A worker sequentially works on jobs. Each time a job is completed, a new one is begun. Each job, independently, takes a random amount of time having distribution F to complete. However, independently of this, shocks occur according to a Poisson process with rate λ. Whenever a shock occurs, the worker discontinues working on the present job and starts a new one. In the long run, at what rate are jobs completed?

empty

2 Exercise (10 points)

A scientist has a machine for measuring ozone in the atmosphere that is located in the mountains just north of Los Angeles. At times of a Poisson process with rate 1, storms or animals disturb the equipment so that it can no longer collect data. The scientist comes every L units of time to check the equipment. If the equipment has been disturbed, then she can usually fix it quickly, so we will assume that the repairs take 0 time. What is the limiting fraction of time the machine is working?

empty

3 Exercise (5+5+4+4 points)

Consider a three station queuing network in which arrivals to servers $i = 1, 2, 3$ occur at rates $3, 2, 1$, respectively. Service at stations $i = 1, 2, 3$ occurs at rates $4, 5, 6$, respectively. Suppose that the probability of going to station j when exiting station i is given by $p_{1,2} = 1/3$, $p_{1,3} = 1/3$, $p_{2,3} = 2/3$, and $p_{i,j} = 0$ otherwise.

1. Describe the network by the its transition graph.

2. Show that the system is stable.

3. Find its stationary distribution

4. Find the average number of customers in the system.

4 Exercise (3+7+7+3 points)

1. For a system $M/M/1$ having arrivals at rate λ and service at rate 2μ, find W.

2. Consider a system $M/M/2$ with arrivals at rate λ and with each server at rate μ.

   (a) Find the stationary distribution.

(b) Compute W.

3. Show that W is smaller in an $M/M/1$ model having arrivals at rate λ and service at rate 2μ than it is in a two-server $M/M/2$ model with arrivals at rate λ and with each server at rate μ.

5 Exercise (7+6+3 points)

A facility produces items according to a Poisson process with rate λ. However, it has shelf space for only k items and so it shuts down production whenever k items are present. Customers arrive at the facility according to a Poisson process with rate μ. Each customer wants one item and will immediately depart either with the item or empty handed if there is no item available.

1. Find the proportion of customers that go away empty handed.

2. Find the average time that an item is on the shelf.

3. Find the average number of items on the shelf.

6 Exercise (7+7 points)

In a bicycle race between two competitors, let $Y(t)$ denote the amount of time (in seconds) by which the racer that started in the inside position is ahead when $100t$ percent of the race has been completed, and suppose that $\{Y(t), 0 \leq t \leq 1\}$ can be effectively modeled as a Brownian motion process with variance parameter σ^2.

1. If the inside racer is leading by σ seconds at the midpoint of the race, what is the probability that she is the winner?

2. If the inside racer wins the race by a margin of σ seconds, what is the probability that she was ahead at the midpoint?

7 Exercise (10 points)

Suppose you own one share of a stock whose price changes according to a standard Brownian motion process. Suppose that you purchased the stock at a price $b + c$, $c > 0$, and the present price is b. You have decided to sell the stock either when it reaches the price $b + c$ or when an additional time t goes by (whichever occurs first). What is the probability that you do not recover your purchase price?

Formula sheet

1. $$\sum_{n=m}^{\infty} x^n = \frac{x^m}{1-x},$$
$$\sum_{n=m}^{\infty} nx^n = \frac{x^m(m+x-mx)}{(1-x)^2},$$
for any integer $m \geq 0$.

2. $$\pi_n = \frac{\theta_n}{\sum_{n=0}^{\infty} \theta_n},$$
for $n = 0, 1, 2, \ldots$, where $\theta_0 = 1$ and $\theta_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}$ for $n \geq 1$.

3. $$\int xe^{ax} dx = \frac{(ax-1)e^{ax}}{a^2} + C.$$

4. Little's formulas $L = \lambda_a W$ and $L_Q = \lambda_a W_Q$.

5. $L_Q = L - 1 + \pi(0)$.

6. For a system $M|M|1$, we have $L = \frac{\lambda}{\mu - \lambda}$.

7. For a network of k stations, we have
   (a) $\pi(n_1, n_2, \ldots, n_k) = \prod_{i=1}^{k} \left(1 - \frac{r_i}{\mu_i}\right) \left(\frac{r_i}{\mu_i}\right)^{n_i}$.
   (b) $L = \sum_{i=1}^{k} \frac{r_i}{\mu_i - r_i}$.

8. Let $(X(t))_t$ be a Brownian motion.
   (a) If $s < t$, we have
   $$E[X(s)|X(t) = B] = \frac{s}{t}B,$$
   $$\text{Var}[X(s)|X(t) = B] = \frac{s}{t}(t-s)$$
   (b) $P\{T_a \leq t\} = \frac{2}{\sqrt{2\pi}} \int_{|a|/\sqrt{t}}^{\infty} e^{-y^2/2} dy$